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Abstract

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MATHEMATICAL PHYSICS

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GROWING SOLUTIONS OF THE SCHRÖDINGER EQUATION

(Presented by Academician V. I. Smirnov on 5 IV 1965)

1. Statement of the problem. Many interesting solutions of the Schrödinger equation $Hu \equiv -\Delta u + v(x)u = \lambda u$ ($x = (x_1, \dots, x_n)$ is a point of Euclidean space E_n, λ is a complex number) with a decreasing potential $v(x)$ are obtained as solutions of integral equations of the form

$$u(x) = u_0(x) - \int \Gamma(x-y, \lambda) v(y) u(y)\, dy, \tag{1}$$

where $u_0(x)$ is a solution of the equation $\Delta u_0 + \lambda u_0 = 0$, and $\Gamma(x, \lambda)$ is the Green function of the Helmholtz equation

$$\Delta\Gamma(x, \lambda) + \lambda\Gamma(x, \lambda) = -\delta(x).$$

Here $\delta(x)$ is the δ-function concentrated at the origin. The integration in (1), and everywhere below unless otherwise specified, is over all of E_n. As u_0 one usually takes plane waves $\exp\{i(k, x)\}$, $k = (k_1, \dots, k_n) \in E_n$, $(k, x) = k_1 x_1 + \dots + k_{nx} n$, with $\lambda = k^2 = (k, k)$. Different choices of the Green function $\Gamma(x, \lambda)$ correspond to different systems of solutions of the equation $Hu = k^2 u$.

The most commonly used Green function is the kernel of the resolvent of the Laplace operator $Tu = -\Delta u$, considered as a self-adjoint operator in $L_2(E_n)$:

$$R(x, \lambda) = (2\pi)^{-n} \int \exp\{i(m, x)\}(m^2 - \lambda)^{-1}\, dm. \tag{2}$$

For fixed x this function is analytic in λ in the entire complex plane with a cut along the positive part of the real axis. The integral equation (1) with $\Gamma(x, \lambda) = R(x, k^2 + i0)$ and $u_0 = \exp\{i(k, x)\}$ is the well-known integral equation of scattering theory. Its rigorous investigation for $n = 3$ was carried out by A. Ya.

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Povzner ([1]), who proved that its solutions form a complete orthonormal system of eigenfunctions of the continuous spectrum of the operator H in $L_2(E_n)$.

In the one-dimensional case the Green functions are no less commonly used:

$$K_+(x,\lambda) = \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}}\,\theta(-x); \qquad K_-(x,\lambda) = -\frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}}\,\theta(-x),$$

where $\theta(a) = 1,\ a > 0;\ \theta(a) = 0,\ a < 0$. For fixed x these are entire functions of λ. The solutions of equation (1) with free term $u_0 = \exp\{\pm i(k+iq)x\}$, $q > 0$, and with the functions $K_\pm(x,(k+iq)^2)$ as $\Gamma(x,\lambda)$, play an important role in the study of the inverse problem of scattering theory by the method of V. A. Marchenko (for a detailed exposition see ([2,3])).

The question arises: is there a natural multidimensional analogue of the Green functions K_+ and K_-? In the present work a set of Green functions for the multidimensional Helmholtz equation is described which, in a certain sense, may be regarded as a generalization of the functions $K_\pm$, and the properties of the corresponding solutions of the Schrödinger equation are investigated.

2. Main proposition.

Let us note that the Green functions K_+ and K_- are also connected with the resolvent of a certain operator. Consider the operator T_q, defined by the differential operator $Tu = -\Delta u$ in the space M_q of functions $u(x)$ that are square-integrable with weight $\exp\{2(q,x)\}$, where q is a fixed vector. The operator T_q is non-self-adjoint and has continuous spectrum. As eigenfunctions one may take $\exp\{i(k+iq,x)\}$. The resolvent kernel of the operator T_q is given by the formula

$$G_q(x,\lambda) = (2\pi)^{-n}\int \exp\{i(m+iq,x)\}[(m+iq)^2-\lambda]^{-1}\,dm. \tag{3}$$

It is not difficult to verify that for $\operatorname{Im}\sqrt{\lambda} > |q|$ this function coincides with $R(x,\lambda)$ from (2).

For $n = 1$, the eigenvalues $(k+iq)^2$ run over the parabola $\operatorname{Im}\sqrt{\lambda} = |q|$. If $\operatorname{Im}\sqrt{\lambda} < |q|$, then the function $G_q(x,\lambda)$ coincides with $K_+(x,\lambda)$ or $K_-(x,\lambda)$, depending on the sign of q.

For $n > 1$, the eigenvalues $(k+iq)^2$ fill the entire interior of the parabola, i.e. the region $\operatorname{Im}\sqrt{\lambda} \le |q|$. In this region the kernels $G_q(x,\lambda)$ depend on λ continuously, but not analytically. We propose to regard the collection of kernels $G_q(x,\lambda)$ for $n > 1$ and $\operatorname{Im}\sqrt{\lambda} \le |q|$ as a multidimensional analogue of the kernels $K_+(x,\lambda)$ and $K_-(x,\lambda)$.

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3. Growing solutions.

Consider the integral equation

$$\varphi_q(x,k) = \exp\{i(k+iq,x)\} - \int G_q(x-y,(k+iq)^2)v(y)\varphi_q(y,k)\,dy. \quad (4)$$

Here the function $G_q(x,\lambda)$ occurs with λ inside the parabola, since $\operatorname{Im}\sqrt{(k+iq)^2} < |q|$. The integral operator

$$A_q(k)\varphi(x) = -\int G_q(x-y,(k+iq)^2)v(y)\varphi(y)\,dy,$$

which enters into this equation, is defined in the class C_q of continuous functions $\varphi(x)$ satisfying the estimate $|\varphi(x)| < K\exp\{-(q,x)\}$. Indeed, from (3) it is clear that the kernel $G_q(x,\lambda)$ can be represented in the form $\exp\{-(q,x)\}\widetilde{G}_q(x,\lambda)$, where $\widetilde{G}_q(x,\lambda)$ is bounded for all $x \neq 0$, and at $x = 0$ has a weak singularity of type $|x|^{-1}$. Therefore, if $v(x)$ decreases sufficiently rapidly, for example if $|v(x)| \leq K(1+|x|)^{-2-\varepsilon}$, $\varepsilon > 0$, then the operator $A_q(k)$ maps C_q into C_q.

The class of functions C_q becomes a Banach space if as the norm one takes $\max\exp\{(q,x)\}|\varphi(x)|$. It can be shown that the operator $A_q(k)$ is completely continuous in this space. The free term of equation (4) obviously belongs to C_q, so that we may consider (4) as a Fredholm equation of the second kind.

Unfortunately, we have not succeeded in investigating in detail the question of the structure of the set of exceptional points k, q for which the homogeneous equation corresponding to (4) has nontrivial solutions. It will be clear from what follows that this set is not very large. Let us note that for sufficiently small $v(x)$ exceptional points are absent altogether.

Solutions of equation (4) satisfy the Schrödinger equation $H\varphi_q = (k+iq)^2\varphi_q$ and grow as $|x| \to \infty$ in the direction forming an acute angle with the vector $-q$. Therefore we call them growing solutions of the Schrödinger equation.

4. Analytic properties of the solutions.

Write the free term of equation (4) in the form

$$\exp\{i(k+iq,x)\} = \exp\{is(x,\gamma)\}\exp\{i(k_\perp,x)\}. \quad (5)$$

Here γ is the unit vector in the direction of q, $\gamma = q|q|^{-1}$; $k_\perp$ is the part of the vector k orthogonal to γ; $k_\perp = k-(k,\gamma)\gamma$, and $s = (k,\gamma)+i|q|$. The function (5) is a plane wave analytically continued ···

into the upper half-plane with respect to the component of the vector k directed along γ. It turns out that the solution $\varphi_q(x,k)$ is also an analytic function of s.

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Let us replace in equation (4) the parameters k and q by γ, $k_\perp$, and s. Note that $G_q(x,(k+iq)^2)$ depends on $k_\perp$ only through $\mu^2 = (k_\perp, k_\perp)$. When k and q vary independently in the space E_n, s, γ, and μ run through, respectively, the upper half-plane, the unit sphere, and the positive half-axis. The functions obtained from $\varphi_q(x,k)$ and $G_q(x,(k+iq)^2)$ after the indicated substitution will be denoted by $\varphi_\gamma(x,s,k_\perp)$ and $C_\gamma(x,s,\mu)$. There is the integral representation

$$C_\gamma(x,s,\mu) = \int D_\gamma(x,\mu,t)\exp\{its\}, \tag{6}$$

where, for example, for $n = 3$,

$$D_\gamma(x,\mu,t) = (2\pi)^{-1}\theta(t-(x,\gamma))\left[\delta(x^2-t^2) - \mu\theta(x^2-t^2)J_1(\mu\sqrt{x^2-t^2})(2\sqrt{x^2-t^2})^{-1}\right].$$

Here $J_1(a)$ is the Bessel function. The integration in (6) is in fact over the finite interval $(x,\gamma) < t < |x|$, so that $C_\gamma(x,s,\mu)$ is an entire function of s, and $\exp\{-is(x,\gamma)\}C_\gamma(x,s,\mu)$ is bounded in the upper half-plane.

Arguments analogous to those given in the preceding paragraph lead to the conclusion that equation (4) can be regarded as an equation of the second kind with a completely continuous operator analytically depending on the parameter s in the upper half-plane. Hence it follows that the solution $\varphi_\gamma(x,s,k_\perp)$ itself, for fixed γ and $k_\perp$, is an analytic function of s throughout the upper half-plane, except for special points of pole type.

5. Triangular transformation operator. The integral representation (6) suggests that the solution $\varphi_\gamma(x,s,k_\perp)$ can be sought in the form

$$\varphi_\gamma(x,s,k_\perp) = \exp\{is(x,\gamma) + i(k_\perp,x)\} + \int_{(x,\gamma)}^{\infty} A_\gamma(x,k_\perp,t)\exp\{its\}\,dt. \tag{7}$$

The integral equation for the kernel $A_\gamma(x,k_\perp,t)$, replacing equation (4) for $\varphi_\gamma(x,s,k_\perp)$, has the form

$$A_\gamma(x,k_\perp,t) = \int D_\gamma(x-y,\mu,t-(y,\gamma))\,v(y)\exp\{i(k_\perp,y)\}\,dy -$$

$$-\int D_\gamma(x-y,\mu,t-u)\,v(y)A_\gamma(y,k_\perp,u)\,dy\,du.$$

This is an equation of Volterra type. Its solution, obtained by successive approximations, admits the estimate

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$$|A_\gamma(x, k_\perp, t)| \leq K \exp\{a[t - (x, \gamma)]\}, \tag{8}$$

where the constant $a > 0$ depends on $v(x)$ and cannot, generally speaking, be replaced by zero. For $\operatorname{Im} s > a$ the expression (7) has meaning. It can be shown that the function $\varphi_\gamma(x, s, k_\perp)$ constructed in this way satisfies equation (4). From the preceding arguments there follows the existence of growing solutions for all sufficiently large $|q|$.

6. The operator H_q. The solutions $\varphi_q(x, k)$ may be regarded as proper functions of the continuous spectrum of the non-self-adjoint operator H_q, defined by the Schrödinger operator H in the space M_q (see Sec. 2). It can be shown that, if the integral equation (4) is uniquely solvable, then the systems $\varphi_q(x, k)$ and $\varphi_{-q}(x, k)$ are biorthogonal. This is not surprising, since the operators H_q and H_{-q} are simply related. The indicated property

obviously holds if $v(x)$ is sufficiently small or $|q|$ is sufficiently large. In these cases the system of solutions $\varphi_q(x, k)$ is complete, so that the operators H_q and T_q are linearly equivalent. In the general case this is not so. The spectrum of the operator H_q outside the parabola can only be discrete and here coincides with the discrete spectrum of the operator H in $L_2(E_n)$. Inside the parabola, besides the spectrum to which the solutions $\varphi_q(x, k)$ correspond, the operator H_q may have an additional spectrum, and, as examples show, even a continuous one.

7. As an application, we give a formula for the kernel $D(x, y, \lambda)$ of the resolvent of the Schrödinger operator H in $L_2(E_n)$ in terms of growing solutions. Let $\operatorname{Im}\sqrt{\lambda}$ be sufficiently large, so that there exists q such that $\operatorname{Im}\sqrt{\lambda} > |q| > a$ from (8). Fix such a q. The resolvents of the operators H and H_q for the indicated λ coincide, so that the formula holds

$$D(x, y, \lambda) = (2\pi)^{-n} \int [(m + iq)^2 - \lambda]^{-1} \varphi_q(x, m) \overline{\varphi_{-q}(y, m)}\, dm. \tag{9}$$

Replacing q and m by γ, s and $m_\perp$, we obtain

$$D(x, y, \lambda) = (2\pi)^{-n} \int (s^2 + m_\perp^2 - \lambda)^{-1} \varphi_\gamma(x, s, m_\perp) \varphi_{-\gamma}(y, s, -m_\perp)\, ds\, dm_\perp.$$

The integration with respect to s is carried out along the line $\operatorname{Im} s = |q|$. For $(x, \gamma) > (y, \gamma)$ the contour can be closed in the upper half-plane. The only singularity of the integrand is the pole at the point $s = \sqrt{\lambda - m_\perp^2}$. As a result we arrive at the formula

$$\begin{aligned} D(x, y, \lambda) = 2\pi i (2\pi)^{-n} \int \varphi_\gamma(x, \sqrt{\lambda - m_\perp^2}, m_\perp) \varphi_{-\gamma}(y, \sqrt{\lambda - m_\perp^2}, -m_\perp) \\ \times \left(2\sqrt{\lambda - m_\perp^2}\right)^{-1} dm_\perp, \end{aligned} \tag{10}$$

valid for $(x, \gamma) > (y, \gamma)$ and $\operatorname{Im} \sqrt{\lambda} > a$. There is reason to believe that it is in fact valid for all λ. It is characteristic that in the right-hand side of (10), in contrast to (9), solutions of the equation $Hu = \lambda u$ with the same λ as appears on the left-hand side take part. Formula (10) may be regarded as a natural generalization of the known representations of the one-dimensional Green' s function in terms of two linearly independent solutions.

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Note: Figure translations are in progress. See original paper for figures.

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