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Mathematics

1965

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Abstract

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Definition of Boundary Elements by Means of Sections

(Presented by Academician M. A. Lavrent' ev, 1 III 1965)

Mathematics

In order to characterize the correspondence of boundaries arising under quasiconformal (^{1,6}) mappings of the ball in n-dimensional Euclidean space, boundary elements were introduced (^{3,4}) (as classes of equivalent regular sequences), in terms of which this correspondence is completely described. In the present note the study of these boundary elements is continued. In particular, it is established that, just as in the planar case (where the boundary elements introduced coincide with prime ends in the sense of Carathéodory), they can be defined by a chain of sections of the domain by hypersurfaces with diameters tending to zero. The latter definition is equivalent to the original one, but is more convenient for recognizing boundary elements.

The concepts and notation used are as follows: $\mu_D(p,q)$ is the lower bound of the diameters of arcs γ from the domain D joining the points $p,q \in D$; $d(M)$ is the diameter of the set M; ∂D is the boundary of D; $\partial_D G = \partial G \cap D$ is the boundary of G relative to D; $\overline{M}$ is the closure of M; $\emptyset$ is the empty set; $D^* = T(D)$ is the image of D under the mapping $y = T(x)$; $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$; E^n is Euclidean space of dimension n. By an arc is meant a homeomorphic image of a segment, and by a path, a homeomorphic image $\gamma : p = p(t)$ of the half-interval $[0 \le t < 1)$. We say that the path γ goes to the point p_1 if $p(t) \to p_1$ as $t \to 1$. We shall also use the following notions, whose definitions may be found in (^{4}): D-equivalent paths; attainable point of the boundary of a domain; regular sequence of subdomains; equivalent sequences of subdomains; boundary element of a domain.

We shall give here only the definition of a regular sequence of subdomains, since it is especially important for what follows.

Definition. A sequence $\{D_n\}$ of subdomains of a domain D is called **regular** if: a) $D_{n+1} \subset D_n$; b) $\bigcap_{n=1}^{\infty} [\overline{D_n}] \subset \partial D$; c) $g_n = \partial_D D_n$, $D_n \in \{D_n\}$, is connected; d) $\mu_D(g_n, g_{n+1}) > 0$ $(n = 1, 2, \dots)$; e) there is at most one attainable boundary point of D which is an attainable boundary point of each of the domains D_n.

Lemma 1. *If a sequence $\{D_n\}$ of subdomains of a domain D satisfies conditions a), b), c), d) of regularity and $d(\partial_D D_n) \to 0$ as $n \to \infty$, then the sequence $\{D_n\}$ is regular.*

To prove this assertion it is enough to verify that $\{D_n\}$ satisfies condition e) of regularity. If condition e) were not satisfied, then in D there would be found two paths γ_1 and γ_2 which determine distinct attainable boundary points (p_i, γ_i) $(i = 1, 2)$ of the domain D, and which are attainable boundary points of each of the domains $D_n \in \{D_n\}$. By virtue of condition b) of regularity, beginning with some m, we shall have $\gamma_i \cap g_n \neq \emptyset$, where $g_n = \partial_D D_n$ $(i = 1, 2)$, $n > m$. But $d(g_n) \to 0$ as $n \to \infty$, and therefore the paths γ_1 and γ_2 go to one and the same point $p = p_1 = p_2$ boundary of the domain D. To complete the proof, it remains for us to verify that in any neighborhood $U(p)$ of the point p the paths γ_1 and γ_2 can be joined by an arc $\lambda \subset D$. Let m be so large that $g_n \subset U(p)$ for $n > m$. Fix some $k > m$. By condition c) of regularity, g_k will be entirely contained in one of the components of $D \cap U(p)$. But $\gamma_i \cap g_k \neq \emptyset$ $(i = 1, 2)$; consequently, in this component there is an arc $\lambda \subset D \cap U(p)$ joining γ_1 and γ_2. Thus the paths γ_1 and γ_2 are equivalent and determine one and the same attainable boundary point. The contradiction obtained completes the proof of Lemma 1.

Consider in a domain $D \subset E^n$ a surface g, homeomorphic to the $(n-1)$-dimensional Euclidean space E^{n-1}.

Definition. We shall say that there is a **section** of the domain D if a surface $g \subset \bar{D}$ is given such that $D \setminus g$ decomposes into two domains D' and D'', for which

$$\partial_D D' = \partial_D D'' = g.$$

Definition. A **chain of sections** of the domain D is a sequence of sections for which the cutting surfaces g_{k-1} and g_{k+1} lie in different components of the section of D by the surface g_k $(k = 2, 3, ...)$.

Definition. A chain of sections is called **defining** if $\mu_D(g_k, g_{k+1}) > 0$ $(k = 1, 2, ...)$ and $d(g_k) \to 0$ as $k \to \infty$.* Each defining chain of sections naturally generates a certain sequence $\{D_k\}$ of subdomains of D nested in one another which, by Lemma 1, will be a regular sequence of subdomains of D, defining a certain boundary element of the domain D. Thus a defining chain of sections generates a boundary element of D. We now show that in this way one can obtain any boundary element of the domain D.

Lemma 2. *Let D be a domain in E^n admitting a quasiconformal mapping onto a half-space, and let $(\Gamma, \{D_n\})$ be an arbitrary boundary element of D. Then there exists a defining chain of sections of the domain D generating the given boundary element.*

Let $y = T(x)$ be a Q-quasiconformal mapping of the half-space $x_n > 0$ onto D, under which the element $(\Gamma, \{D_n\})$ corresponds to the boundary point $x = 0$ of

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* In order not to pass to the spherical metric, throughout here D is understood to be a bounded domain.

the initial half-space R. Consider the layer $a < |x| < b$, $x_n > 0$. As is known (^{2,5}), in this layer there is a hemisphere $S: \; |x| = r$, $x_n > 0$, $a < r < b$, for the diameter of whose image S^* the estimate

$$d(S^*) \leq C(n) \left(\frac{V(D)}{\ln b/a} \right)^{1/n} Q$$

holds, where $V(D)$ is the volume of the domain $\dot{D}$ (see the footnote on p. 736), and $C(n)$ is a constant depending only on the dimension of the space. From this estimate follows the possibility of choosing a sequence of hemispheres $S_m: \; |x| = r_m$, $x_n > 0$, $r_m \to 0$ as $m \to \infty$, for which $d(S_m^*) \to 0$ as $m \to \infty$. Consider now the sequence $\{B_m\}$ of half-balls $B_m: \; |x| < r_m$, $x_n > 0$. Then $S_m = \partial_R B_m$. The sequence $\{B_m\}$ is a regular sequence of subdomains of R corresponding to the boundary point $x = 0$. The image $\{B_m^*\}$ in D under the quasiconformal mapping $y = T(x)$ is a regular sequence (^{3}) $\{B_m^*\}$ of subdomains of D, belonging to the class of equivalent regular sequences of the original boundary element $(\Gamma, \{D_n\})$. Thus $\{B_m^*\}$ defines the boundary element $(\Gamma, \{D_n\})$. But, by construction, $\{B_m^*\}$ is generated by a defining chain of sections of the domain D, produced by the surfaces S_m^*. Consequently, for the given boundary element of the domain a defining chain of sections has been selected which generates this boundary element. Lemma 2 is proved.

* If the surfaces g_k, generally speaking, are not homeomorphic to E^{n-1}, then the corresponding sections, chains of sections, and defining chains of sections will be called **general**.

The following is the union of these lemmas.

Theorem 1. *If a domain $D \subset E^n$ $(n \geq 2)$ admits a quasiconformal mapping onto a ball, then every boundary element D is generated by some defining chain of sections of the domain D, and every such chain of sections generates some boundary element D.*

We shall now show that, if there exists some defining chain of sections generating a boundary element of a domain, one can construct a common defining chain of sections which generates the same boundary element, but whose cutting surfaces lie on some fixed system of surfaces. In particular, as such a system one may take, for example, a family of concentric spheres contracting to the center, or an analogous family of surfaces of hypercubes.

Definition. A point $q \in \partial D$ is called **essential** if there exists a defining chain of sections D whose cutting surfaces contract to q.

By virtue of Theorem 1, on every continuum

$$\Gamma = \bigcap_{n=1}^{\infty} \overline{D}_n \subset \partial D,$$

corresponding to some boundary element $(\Gamma, \{D_n\})$, there is at least one essential point.

Theorem 2. *Let a domain $D \subset E^n$ admit a quasiconformal mapping onto a ball; let $\{D_n\}$ be a regular sequence of subdomains of D, for which $\partial_D D_n \to q \in \partial D$ as $n \to \infty$; let $\{B_k\}$ be an arbitrary sequence of Jordan domains containing q and contracting to q as $k \to \infty$. Then there exists a regular sequence $\{G_m\}$ of subdomains of D, defining the same boundary element as the sequence $\{D_n\}$, for which $\partial_D G_m \subset \partial B_{k_m}$.*

First of all, from the sequence $\{B_k\}$ we select a subsequence $\{B_{k_m}\}$ satisfying the following conditions:

$$\overline{B}_{k_{m+1}} \subset B_{k_m} \quad (m = 1, 2, ...)$$

and

$$D \setminus \overline{B}_{k_1} \neq \emptyset.$$

Fix some point $p \in D \setminus \overline{B}_{k_1}$. The desired regular sequence is constructed as follows.

Take the first domain $D_{n_1} \in \{D_n\}$ such that $\partial_D D_{n_1} \subset B_{k_1}$ and $p \notin D_{n_1}$. If A_1 is the component of $D \setminus \overline{B}_{k_1}$ containing the point p, then $D_{n_1} \cap A_1 = \emptyset$, since otherwise one could join p with some point from D_{n_1} without intersecting B_{k_1}, and hence also $\partial_D D_{n_1}$. But this contradicts the fact that $p \notin D_{n_1}$. We now take as G_1 the component of $D \setminus \overline{A}_1$ containing $\partial_D D_{n_1}$. Then $G_1 \supset D_{n_1}$.

Similarly, for B_{k_m} we find a domain D_{n_m}, distinct from those already used, such that

$$\partial_D D_{n_m} \subset B_{k_m}.$$

We find A_m—the component of $D \setminus \overline{B}_{k_m}$ containing the point p; and, finally, as G_m we take the component of $D \setminus \overline{A}_m$ containing $\partial_D D_{n_m}$. As in the case G_1, we verify that

$$G_m \supset D_{n_m}.$$

We shall show that the sequence $\{G_m\}$ is the desired one.

Since $B_{k_{m+1}} \subset B_{k_m}$, we have $A_{m+1} \supset A_m$, and, since $G_m \cap G_{m+1} \neq \emptyset$, we have

$$G_{m+1} \subset G_m \quad (m = 1, 2, ...).$$

Condition a) of regularity has been verified. Condition d) is also fulfilled, for

$$\mu_D(\partial_D B_{k_m}, \partial_D B_{k_{m+1}}) > 0,$$

while

$$\partial_D G_m \subset \partial B_{k_m} \quad (m = 1, 2, ...),$$

therefore

$$\mu_D(\partial_D G_m, \partial_D G_{m+1}) > 0.$$

Next,

$$\Gamma = \bigcap_{m=1}^{\infty} \overline{G}_m \subset \partial D.$$

Indeed, if $E = \Gamma \cap D \neq \emptyset$, then we join by an arc $\gamma \subset D$ the point p with an arbitrary point of E. Since $g_m = \partial_D G_m$ contract to the point $q \in \partial D$ as $m \to \infty$, for all sufficiently large m we shall have $g_m \cap \gamma = \emptyset$. But this contradicts the fact that $g_m = \partial_D G_m$, since $p \notin G_m$. Thus condition b) of regularity is also fulfilled.

Taking into account that $d(g_m) \to 0$ as $m \to \infty$, in order to prove the regularity of the sequence $\{G_m\}$, by Lemma 1 it remains only to verify the connectedness of g_m $(m = 1, 2, ...)$. This will at the same time prove that the g_m generate a common defining chain of sections, since it is easy to see–

that g_m divides D into exactly two subdomains: G_m and $H_m = D \setminus \overline{G}_m$. Let us note here also that $g_m \subset \partial B_{k_m}$; therefore every point of g_m is an attainable boundary point both of G_m and of H_m. Thus, the connectedness of ξ_m $(m = 1, 2, ...)$ establishes the following

Lemma 3. Let A and B be subdomains of a domain $D \subset E^n$ having a common relative boundary

$$\Gamma = \partial_D A = \partial_D B,$$

each point of which is attainable both from A and from B. If the domain D is simply connected, then the set Γ is connected.

Suppose that Γ is not connected. Let Γ_1 be a connected component of Γ, and let $\Gamma_2 = \Gamma \setminus \Gamma_1$. By assumption, $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$. Consider two arcs γ_1 and γ_2, all points of which, except for their common endpoints $a \in \Gamma_1$ and $b \in \Gamma_2$, lie respectively in A and B. Since $\pi_1(D) = 0$, there exists a homotopic deformation in D,

$$F_\alpha(t) \quad [0 \leq \alpha \leq 1,\ 0 \leq t \leq 1],$$

of the arc

$$\gamma_1 = F_0(t) \quad [0 \leq t \leq 1]$$

into

$$\gamma_2 = F_1(t) \quad [0 \leq t \leq 1],$$

under which the endpoints $a = F_\alpha(0)$ and $b = F_\alpha(1)$ remain fixed.

It is clear that the sets F_1 and F_2 of points of the deformation square

$$K\ [0 \leq \alpha \leq 1,\ 0 \leq t \leq 1],$$

whose images under the mapping $F_\alpha(t)$ lie respectively on Γ_1 and Γ_2, are closed in K and do not intersect. The endpoints $a \in \Gamma_1$ and $b \in \Gamma_2$ of the initial arc γ_1 remain fixed during the deformation; therefore the lateral sides

$$f_1\ [t = 0,\ 0 \leq \alpha \leq 1]$$

and

$$f_2\ [t = 1,\ 0 \leq \alpha \leq 1]$$

of the deformation square belong respectively to F_1 and F_2. In the (α, t)-plane consider a closed Jordan curve γ which separates the connected component F_1 containing f_1 from the corresponding component F_2 containing f_2, and intersects neither F_1 nor F_2. The existence of the curve γ in the case when F_1 and F_2 have only a finite number of connected components is obvious. In the general case it is enough to consider a small δ-neighborhood of each of the sets F_1 and F_2, which will already contain a finite number of connected components. Let λ be an arc on γ, lying in the deformation square and joining interior points of the upper and lower bases of the square. Then the image of λ under the mapping $F_\alpha(t)$ has common points both with A and with B, and at the same time does not intersect $\Gamma = \Gamma_1 \cup \Gamma_2$. The contradiction obtained completes the proof of Lemma 3, and with it that of Theorem 2.

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Received
10 II 1965

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Note: Figure translations are in progress. See original paper for figures.