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Abstract

Full Text

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Boundary Estimates for the Solution of the Problem with an Oblique Derivative for a Parabolic Equation in a Noncylindrical Domain

(Presented by Academician S. L. Sobolev on 30 VI 1964)

In our work ([5]), a priori estimates were obtained in Hölder norms for the solution of the III boundary-value problem (with a conormal derivative) for a general parabolic equation of the 2nd order in a noncylindrical domain. The method of these papers makes it possible to obtain a priori estimates in Hölder norms for the solution of the problem with an oblique derivative for a parabolic equation of the 2nd order and for the solution of general boundary-value problems for parabolic systems in the sense of I. G. Petrovskii, satisfying a solvability condition of the Lopatinskii type.

In the present paper we consider the parabolic equation of the 2nd order

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)u-\frac{\partial u}{\partial t}=f(x,t), \quad (x,t)\in Q, \quad (1)$$

with the initial condition

$$u(x,0)=\psi(x), \quad x\in\Omega=\overline{Q}\cap\{t=0\} \quad (2)$$

and the boundary condition

$$\sum_{i=1}^{n}\nu_i(x,t)\frac{\partial u(x,t)}{\partial x_i}+\mu(x,t)u(x,t)=\varphi(x,t), \quad (x,t)\in\Gamma, \quad (3)$$

where Q is a domain (possibly unbounded in x_i) in the space of variables $(x,t) = (x_1, x_2, \dots, x_n; t)$, lying between the hyperplanes $t=0$ and $t=T>0$; Γ is the lateral surface of the domain Q. If $\gamma_i(x,t)$ $(i=1,2,\dots,n)$ are the cosines of the angles formed by the inward normal to Γ_t (at the point (x,t)) with the axes ox_i, where Γ_t is the section of the surface Γ by the hyperplane $t=\text{const}$, then

$$\sum_{k=1}^{n}\nu_k(x,t)\gamma_k(x,t)\geq\nu_0>0, \quad \nu_0=\text{const}.$$

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A_1. The lateral surface Γ has at each point a tangent plane nowhere orthogonal to the axis ot. For each point $P(x,t) \in \Gamma$ there exists an $(n+1)$-dimensional sphere $S_\delta(P)$ with center at the point P and radius $\delta > 0$ (δ does not depend on the choice of the point P on Γ) such that the part of Γ lying in the sphere $S_\delta(P)$ can, for some i ($1 \le i \le n$), be represented in the form

$$x_i = h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n; t),$$

where the function h has a first derivative with respect to t and second derivatives with respect to x_k, satisfying Hölder conditions in x and t with exponents α and $\alpha/2$, respectively. Here and below $0 < \alpha < 1$. In this case we shall say that the function $h(x,t)$ belongs to the class $C^{2+\alpha,\,1+\alpha/2}_{x,t}$ (has bounded norm $|h|_{2+\alpha}$, cf. ([5])).

A$_2$. Equation (1) is uniformly parabolic in Q, i.e., for any real vector ξ and for all $(x,t) \in \overline{Q}$

$$M_1 \sum_{i=1}^{n} \xi_i^2 \ge \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge M_0 \sum_{i=1}^{n} \xi_i^2,$$

where $M_i > 0$ ($i = 0, 1$) are constants.

A$_3$.

$$|a_{ij}|_\alpha^Q + |b_i|_\alpha^Q + |c|_\alpha^Q \le M_2, \qquad |f|_\alpha^Q < +\infty.$$

A$_4$.

$$|\nu_i|_{1+\alpha}^\Gamma + |\mu|_{1+\alpha}^\Gamma \le M_2, \qquad |\varphi|_{1+\alpha}^\Gamma < +\infty, \qquad |\psi|_{2+\alpha}^\Omega < +\infty.$$

A$_5$. The functions f, ψ, φ, μ, and ν_i are compatible, by virtue of equation (1), on the edge $\Gamma \cap \Omega$.

Consider in the strip

$$D_T = \{(x,t),\ |x_i| < +\infty,\ i = 1, 2, \dots, n-1;\ 0 < x_n < +\infty,\ 0 < t < T < +\infty\}$$

the boundary-value problem for the heat-conduction equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = f(x,t), \qquad (x,t) \in D_T; \tag{4}$$

$$u(x,0) = \psi(x), \qquad x \in \Omega_0 = \overline{D_T} \cap \{t = 0\}; \tag{5}$$

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$$\frac{\partial u(x,t)}{\partial l_0} = \varphi(x,t), \qquad (x,t) \in \Gamma_0 = \overline{D_T} \cap \{x_n = 0\}, \tag{6}$$

where l_0 is a constant direction lying in the plane passing through the point (x,t) orthogonally to the ot axis and forming an acute angle with the axis ox_n.

Lemma 1. Let the function $f(x,t)$ be defined in D_T and $|f|^{D_T} < +\infty$, let the function $\psi(x)$ be defined in Ω_0 and $|\psi|_{2+\alpha}^{\Omega_0} < +\infty$, and let the function $\varphi(x_1, \dots, x_{n-1}, t)$ be defined on Γ_0 and $|\varphi|_{1+\alpha}^{\Gamma_0} < +\infty$, with f, ψ, and φ compatible on the edge $\Gamma_0 \cap \Omega_0$. Then there exists a unique (in the class of bounded functions continuous on $\overline{D_T}$) solution $v(x,t)$ of problem (4)–(6), for which the inequalities

$$[v]_{2+\alpha}^{D_T} \le C(M_0, M_1, l_0)(|f|_{\alpha}^{D_T} + [\psi]_{2+\alpha}^{\Omega_0} + [\varphi]_{1+\alpha}^{\Gamma_0}),$$

$$[v]_0^{D_T} \le C(M_0, M_1, T, l_0)([f]_0^{D_T} + [\psi]_0^{\Omega_0} + [\varphi]_0^{\Gamma_0}).$$

It is sufficient to prove Lemma 1 for the one-dimensional heat-conduction equation with zero initial condition (see ([1],[5],[6])). In this case the solution of the problem under consideration in D_T can be represented explicitly in the form (for brevity of notation we put $n = 2$)

$$v(x,t) = -\frac{1}{2\pi} \int_0^t d\tau \int_{-\infty}^{+\infty} H(x_1 - \xi_1, x_2, t - \tau) \varphi(\xi_1, \tau)\, d\xi_1, \tag{7}$$

where the half-space kernel H has the form

$$\begin{aligned} H(x_1 - \xi_1, x_2, t - \tau) = {} & \frac{a_2}{t-\tau} \exp\left\{-\frac{(x_1-\xi_1)^2 + x_2^2}{4(t-\tau)}\right\} \\ & - \frac{a_1[a_1 x_2 - a_2(x_1 - \xi_1)]}{(t-\tau)^{3/2}} \exp\left\{-\frac{[a_1 x_2 - a_2(x_1-\xi_1)]^2}{4(t-\tau)}\right\} \\ & \times \left[\Phi\left(\frac{a_1(x_1-\xi_1) + a_2 x_2}{2(t-\tau)^{1/2}}\right) - \frac{\sqrt{\pi}}{2}\right]. \end{aligned} \tag{8}$$

Here (a_1, a_2) are the direction cosines of the direction l_0,

$$\Phi(x) = \int_0^x \exp\{-y^2\}\, dy.$$

The solution (7) was obtained by us by applying the Fourier transform with respect to x_i $(i = 1, 2, \dots, n-1)$ and the Laplace transform with respect to t.

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A kernel of the type (8) is also given in the paper ([2]) for arbitrary cylindrical domains. By estimating the solution (7) directly, we obtain the estimates of Lemma 1. The uniqueness of the representation (7) is established with the aid of the paper ([3]).

Bearing in mind the interior a priori estimates and the a priori estimates near the base Ω, the estimates in the whole noncylindrical domain Q of the solution of (1)–(3) are established with the aid of Lemma 1 by the method of localization.

Theorem 1. *Let $Q_1 \subseteq Q$ be a subdomain of Q (in particular, it may coincide with Q), with $\Omega_1 = \overline{Q}_1 \cap \Omega$, $\Gamma_1 = \overline{Q}_1 \cap \Gamma$. Suppose that conditions A_1–A_5 are fulfilled in the domains Q, Ω_1, Γ_1, respectively, and that $u(x,t)$ is a solution, bounded in $\overline{Q}$, of problem (1)–(3), where Ω in (2) and Γ in (3) are replaced respectively by Ω_1 and Γ_1. Suppose, finally, that*

$$|u|_{2+\alpha}^{Q\cup\Gamma_1\cup\Omega_1} < +\infty.$$

Then

$$|u|_{2+\alpha}^{Q_1} \leq C(Q_1, Q, M_0, M_1, M_2, \nu_0, \delta)(|f|_\alpha^Q + |\varphi|_{1+\alpha}^\Gamma + |\psi|_{2+\alpha}^\Omega + [u]_0^Q),$$

where $[u]_0^Q = \sup_Q |u|$.

Remark. Increasing the requirements on the smoothness of the surface Γ, the coefficients of equation (1), the functions appearing in (2), (3), and the order of compatibility leads to a corresponding increase in the smoothness of the solution of problem (1)–(3).

Theorem 2. *In the case of a bounded domain Q, the estimate*

$$|u|_{2+\alpha}^Q \leq C(|f|_\alpha^Q + |\varphi|_{1+\alpha}^\Gamma + |\psi|_{2+\alpha}^\Omega)$$

is valid, where the constant C depends on equation (1) and on the domain Q, but does not depend on the function $u(x,t)$.

Using refined investigations of the properties of heat potentials (see ([4])), by the method of continuation in a parameter, with the aid of Theorem 2 one can prove the following existence theorem.

Theorem 3. *Let the surface Γ be of type $J_{1,1,(1+\beta)/2}^{1,\beta,\beta/2}$ $(0 < \alpha < \beta < 1)$, where $\beta > 0$ is arbitrary (see ([4])). Suppose that conditions A_2–A_5 are fulfilled.*

Then there exists a solution $u(x,t)$ of problem (1)–(3), having in $\overline{Q}_T$ a first derivative with respect to t and second derivatives with respect to x_i, satisfying Hölder conditions in x and t with exponents α and $\alpha/2$, respectively.

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Note: Figure translations are in progress. See original paper for figures.

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