ON REPRESENTATIONS OF THE MODULAR GROUP ON SPACES OF CUSP FORMS

MATHEMATICS

1965

SovietRxiv

Abstract

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UDC 517-77

MATHEMATICS

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ON REPRESENTATIONS OF THE MODULAR GROUP ON SPACES OF CUSP FORMS

(Presented by Academician Yu. V. Linnik on 15 IV 1965)

Let $\Gamma' = SL(2,\mathbb{Z})$ be the homogeneous modular group, and let $G' \subset \Gamma'$ be a normal subgroup of finite index; let Γ, G be the corresponding groups of fractional-linear transformations. Denote by $\mathfrak{M}_{2k}(G)$ the space of all cusp forms of weight $2k$ (k an integer, $k \geq 1$) with respect to the group G (^{1}).

Then for any element $\sigma \in \Gamma$, $\sigma : z \mapsto (\alpha z + \beta)/(\gamma z + \delta)$, and any function $F(z) \in \mathfrak{M}_{2k}(G)$, the function

$$F(z) \circ \sigma = (\gamma z + \delta)^{2k} F\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)$$

again belongs to the space $\mathfrak{M}_{2k}(G)$. Denote the resulting representation of the finite group Γ/G on the space $\mathfrak{M}_{2k}(G)$ by $\Omega_{2k}(\Gamma/G)$. A description of the representation $\Omega_{2k}(\Gamma/G)$ is important for a number of questions in the theory of modular forms and, in particular, for the theory of Hecke operators.

Theorem. Let G be a normal subgroup of finite index $\mu > 3$ of the inhomogeneous modular group Γ; let χ be a simple nonprincipal character of the group Γ/G; and let $r_{2k}(\chi)$ be the multiplicity of the irreducible representation of the group Γ/G with character χ in the representation $\Omega_{2k}(\Gamma/G)$ of this group on the space of all cusp forms of weight $2k$ ($k \geq 1$) with respect to the group G. Then

$$r_{2k}(\chi)+r_{2k}(\bar{\chi}) = \frac{\chi(1)(2k-1)}{6} - \frac{1}{N}\sum_{n \bmod N} \chi(\tau_0^n) + \frac{(-1)^k}{2}\chi(\sigma_0) + \frac{1}{3}\left(\frac{k+1}{3}\right)(\chi(\sigma_0\tau_0)+\chi((\sigma_0\tau_0)^2)), \tag{1}$$

where $\bar{\chi}$ is the character conjugate to χ; $\tau_0 : z \mapsto z+1$; $\sigma_0 : z \mapsto -1/z$; N is the least natural number such that $\tau_0^N \in G$; $\left(\frac{k+1}{3}\right)$ is the Legendre symbol.

Previously formula (1) was obtained by Hecke for $k = 1$ (^{2}). We outline the proof of the theorem. Denote by M_n the representation of the group Γ' on the space of homogeneous polynomials of degree n in two variables with complex coefficients ($M_0 \equiv 1$) (^{3}). For even n the representation M_n is trivial on the center of the group Γ' (consisting of $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$), so that in this case we actually obtain a representation of the group Γ, which we shall denote by the same letter.

Lemma 1. Let χ_n be the character of the representation M_n; then, in the notation of the theorem, for any integer $k \geq 0$,

$$\chi_{2k}(\sigma_0) = (-1)^k,$$

$$\chi_{2k}(\sigma_0\tau_0) = \chi_{2k}((\sigma_0\tau_0)^2) = -\left(\frac{k+2}{3}\right),$$

where $\left(\frac{k+2}{3}\right)$ is the Legendre symbol.

Proof. The assertions of the lemma are easily derived if one takes into account that $\sigma_0^2 = (\sigma_0\tau_0)^3 = 1$, and uses the known relations between the proper numbers of finite-order elements in the representations M_1 and M_n.

Lemma 2. Let A be a complex matrix of order s such that $A^N = E_s$ (E_s is the identity matrix of order s). Then the number of linearly independent solutions X of the system of linear equations

$$(E_{(2k+1)s} - M_{2k}(\tau_0) \otimes A)X = 0 \tag{2}$$

does not depend on k and is equal to

$$\frac{1}{N} \sum_{n \bmod N} \operatorname{tr}(A^n).$$

Proof. The number of linearly independent solutions of system (2) is equal to the dimension of the space of proper vectors of the operator $M_{2k}(\tau_0) \otimes A$ in the module $U \otimes V$ (where U is the module of the representation M_{2k}, V is the complex module of dimension s in which the operator given, in some basis, by the matrix A acts) belonging to the proper number 1, and is found by direct computation if, as a basis of U, one takes monomials of the form $u^{2k-l}v^l$ ($l = 0, \dots, 2k$), as a basis of V the proper vectors ω_i ($i = 1, 2, \dots, s$) of the operator A, and as a basis of $U \otimes V$ elements of the form $u^{2k-l}v^l \otimes \omega_i$ ($l = 0, \dots, 2k;\ i = 1, 2, \dots, s$).

Let R be a complex representation of the group Γ/G with character χ. Put $M_{2k,\chi} = M_{2k} \otimes R$ and consider the group of parabolic cohomologies $H(\Gamma, M_{2k,\chi})$ of the group Γ relative to the representation $M_{2k,\chi}$ (^{3,4}). Recall that the group H is defined analogously to the ordinary one-dimensional cohomology group of the group Γ with coefficients in the representation module $M_{2k,\chi}$, with the sole difference that an additional condition is imposed on cocycles: for every cocycle $\mathbf{r}$ and every parabolic substitution $\tau \in \Gamma$ there is a vector $\mathbf{a}$ from the representation module $M_{2k,\chi}$ for which

$$\mathbf{r}(\tau) = \mathbf{a} - M_{2k,\chi}(\tau)\mathbf{a}.$$

Lemma 3. For any representation R of the group Γ/G with character χ, in the notation introduced above,

$$\dim_C H(\Gamma, M_{2k,\chi}) \geq \frac{\chi(1)(2k+1)}{6} - \frac{1}{N}\sum_{n \bmod N} \chi(\tau_0^n) + \frac{(-1)^{k+1}}{2}\chi(\sigma_0) + \frac{1}{3}\left(\frac{k+2}{3}\right)(\chi(\sigma_0\tau_0) + \chi((\sigma_0\tau_0)^2)).$$

Proof. Let $H(\Gamma, M_{2k,\chi}, \tau)$ (where τ is an arbitrary parabolic substitution from Γ) be the group defined in the same way as $H(\Gamma, M_{2k,\chi})$, with the additional condition that all cocycles and coboundaries vanish on τ. It is easy to see that $H(\Gamma, M_{2k,\chi}, \tau)$ and $H(\Gamma, M_{2k,\chi})$ are canonically isomorphic. In particular,

$$\dim_C H(\Gamma, M_{2k,\chi}) = \dim_C H(\Gamma, M_{2k,\chi}, \tau_0),$$

and it is enough to estimate the latter dimension. As is known (^{1}), the group Γ is a group with two generators σ_0, τ_0 and defining relations

$$\sigma_0^2 = (\sigma_0\tau_0)^3 = 1. \tag{3}$$

Thus, each cocycle from $H(\Gamma, M_{2k,\chi}, \tau_0)$ is uniquely determined by the vector $X = \mathbf{r}(\sigma_0)$ (since $\mathbf{r}(\tau_0) = 0$). Denote by V the set of solutions X of the system of homogeneous equations

$$[E_{(2k+1)\chi(1)} + M_{2k,\chi}(\sigma_0)]X = 0, \tag{4}$$

$$[E_{(2k+1)\chi(1)} + M_{2k,\chi}(\sigma_0\tau_0) + M_{2k,\chi}((\sigma_0\tau_0)^2)]X = 0 \tag{5}$$

and by W the set of vectors of the form

$$[E_{(2k+1)\chi(1)} - M_{2k,\chi}(\sigma_0)]Y,$$

where

$$[E_{(2k+1)\chi(1)} - M_{2k,\chi}(\tau_0)]Y = 0. \tag{6}$$

(Here and below $M_{2k,\chi}(\sigma)$, for $\sigma \in \Gamma$, denotes the matrix of the operator σ in an arbitrary, but fixed, basis of the representation module $M_{2k,\chi}$.) Then, by virtue of relations (3) and the remark made above,

$$\dim H(\Gamma, M_{2k,\chi}) = \dim H(\Gamma, M_{2k,\chi,\tau_0}) = \dim V - \dim W. \tag{7}$$

Using Lemma 1 and relation (3), we find that the rank of system (4) is equal to

$$r_1 = \frac{1}{2}[(2k+1)\chi(1) + (-1)^k \chi(\sigma_0)],$$

and the rank of system (5) is equal to

$$r_2 = \frac{1}{3}\left[(2k+1)\chi(1) - \left(\frac{k+2}{3}\right)(\chi(\sigma_0\tau_0) + \chi((\sigma_0\tau_0)^2))\right].$$

Since the rank of the system of equations (4)–(5) does not exceed $r_1 + r_2$, we have

$$(2k+1)\chi(1) - \dim V \leq r_1 + r_2,$$

whence, taking account of relations (7), we obtain

$$(2k+1)\chi(1) - \dim H(\Gamma, M_{2k,\chi}) \leq \dim W + r_1 + r_2.$$

Substituting in the last inequality the values of r_1 and r_2, and taking into account that $\dim W$ does not exceed the number of linearly independent solutions of system (6), which was computed in Lemma 2, we obtain the assertion of the lemma.

We now pass directly to the proof of the theorem. As follows from a result of Gunning [4] (a corollary to Theorem 5), in the case when χ is a simple nonprincipal character of the group Γ/G,

$$r_{2k}(\chi) + r_{2k}(\bar{\chi}) = \dim H(\Gamma, M_{2k-2,\chi}).$$

From this relation, on the basis of Lemma 3, we obtain the inequality

$$r_{2k}(\chi) + r_{2k}(\bar{\chi}) \geq \frac{1}{6}(2k-1)\chi(1) - \frac{1}{N}\sum_{m \bmod N} \chi(\tau_0^m) + \frac{(-1)^k}{2}\chi(\sigma_0) +$$

$$+\frac{1}{3}\left(\frac{k+1}{3}\right)(\chi(\sigma_0\tau_0) + \chi((\sigma_0\tau_0)^2)). \tag{8}$$

Multiply both sides of this inequality by $\chi(1) = \bar{\chi}(1)$ and sum over all simple nonprincipal characters χ of the group Γ/G. The left-hand side of the resulting inequality is, evidently,

$$2\sum_{\chi\neq 1}\chi(1)r_{2k}(\chi) = 2(\dim\mathfrak{R}_{2k}(G) - \dim\mathfrak{R}_{2k}(\Gamma))$$

and can be computed explicitly on the basis of the known formulas for the dimensions of spaces of entire modular forms and spaces of Eisenstein series [1], [5]. The right-hand side of the resulting inequality is easily computed on the basis of the known orthogonality relations between the characters of a finite group [6]. The corresponding computations show that the left- and right-hand sides are equal, so that equality holds in the inequalities (8) for each simple χ, $\chi \neq 1$. The theorem is proved.

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Received
13 IV 1965

REFERENCES

[1] R. K. Ganning, *Sborn. per. Matematika*, **8**, 6, 4 (1964).
[2] E. Hecke, *Math. Ann.*, **116**, 469 (1939).
[3] G. Shimura, *J. Math. Soc. Japan*, **11**, No. 4, 291 (1959).
[4] R. C. Gunning, *Trans. Am. Math. Soc.*, **100**, No. 1, 44 (1961).
[5] E. Hecke, *Abh. Math. Seminar Hamburger Univ.*, **5**, 199 (1927).
[6] M. Hall, *The Theory of Groups*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.

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