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Abstract

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ON THE PERIODIC PROBLEM OF THE CALCULUS OF VARIATIONS

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1. Let M be a closed differentiable manifold, and let $J(l)$ be a positive regular functional on M. As is known, on M there exists at least one closed extremal of the functional J (^{3,6}). In this note the case of a **reversible** functional is considered, i.e., one invariant under change of direction on the curve l; such is, in particular, the length functional in a Riemannian metric. We shall call the functional J **nondegenerate** if all closed extremals of J are nondegenerate in the sense of Morse (^{4}). Denote by g^n the curve obtained by traversing the closed extremal g n times $(n = \pm 1, \pm 2, ...)$; we shall call a closed extremal h **simple** if there is no such closed extremal g that $h = g^n$ $(n \geq 2)$. Closed extremals h_1 and h_2 are called **geometrically distinct** if there is no such g that $h_1 = g^m,\ h_2 = g_n$, $(m, n$ are integers). The existence of geometrically distinct closed extremals has so far been proved only under special topological and metric restrictions (the metric condition of Morse in the case where M is diffeomorphic to a sphere). In the general case the possibility was not excluded that all closed extremals on M are iterates of one (cf., for example, (^{2})). In this note we prove

Theorem 1. *A positive regular, reversible, nondegenerate periodic problem of the calculus of variations on a closed manifold has at least two geometrically distinct closed extremals.*

In the case of a simply connected M a more precise assertion is true:

Theorem 2. *Let M be closed and simply connected, and let the first nontrivial homology group of M have dimension j. Then on M there exist geometrically distinct simple closed extremals g_1, g_2, whose Morse indices satisfy the relation $\lambda(g_1) + 1 = \lambda(g_2) \leq j$.*

Since the case of a non-simply-connected M is considered in our work (^{7}), Theorem 4, in what follows M is assumed to be simply connected.

2. Denote by $P,\ \bar{P}$ the spaces of closed oriented (respectively, nonoriented) curves on M, and by L the space introduced in (^{6}), p. 288, of closed oriented curves with a marked point on M. The set of one-point curves in each of these spaces will be denoted by O. The set $(J(l) \leq c)$ in the space C is denoted by J_c^C. We consider the singular homology groups

$$H_i(X \bmod Y,\ A),$$

where the coefficient domain A is the group of integers Z, the cyclic group of order p, Z_p, or the group of rational numbers R.

The i-th type number of a closed extremal h in the space C over the coefficient field A is denoted by

$$m^i_{C,A}(h).$$

3. Let a closed extremal g have index q. Then for every g^n

$$m^{q+1}_{P,R}(g^n) = 0.$$

For the proof, note that, according to Bott (^{1}), p. 172, Theorems A, C), the index

$$\lambda(g^n) = \Sigma\Lambda(\omega),$$

where ω runs through all roots of degree n of 1 (from the simple connectedness of M follows the orientability of g), and $\Lambda(z)$, $|z| = 1$, is an integer-valued function possessing the property

$\Lambda(\bar{z}) = \Lambda(z)$. If $\lambda(g^2) - \lambda(g^1) = \lambda(-1)$ is odd, then by Theorem 21 of A. S. Schwarz (^{8}) all $m_{\bar{P},R}(g^n) = 0$. If, however, $\Lambda(-1)$ is even, then from $\Lambda(\bar{z}) = \Lambda(z)$ for any n there follows the evenness of $\lambda(g^n) - \lambda(g^1)$, and our assertion follows from the same Theorem 21 (^{8}).

4. Suppose that on M there are no closed extremals of index r. Then the group

$$H_r(J^{\bar{P}}_{c_2} \bmod J^{\bar{P}}_{c_1}, R) = 0.$$

It is enough to prove this assertion in two cases: I —when c_2 is a critical value J, $c_1 = c_2 - \varepsilon$, where ε is sufficiently small; II —the segment $[c_1, c_2]$ contains no critical values of J. In case I the type number of the corresponding extremal is equal to zero, and the assertion is proved by means of the standard Morse deformations ((^{4}) or (^{8}), Ch. V). In case II one can first apply to the space P the same arguments as in (^{8}), p. 41, and then, considering the involution $\vartheta(l) = l^{-1}$ on P, apply Conner' s theorem ((^{8}), p. 11).

5. Let g be a closed extremal of least index m on M. Then for $0 \leq i \leq m$, $\pi_i(M) = 0$, and for $0 \leq i \leq m-1$, $\pi_i(J^L_a) = 0$, where a is an arbitrary number smaller than $J(g)$. Let j be the least value of i for which $\pi_i(M) \neq 0$. By hypothesis, $j \geq 2$; assume that $j \leq m$. By the Hurewicz theorem, $H_j(M, Z) \neq 0$. From the formula of universal coefficients there follows the existence of such a prime number p that $H_j(M, Z_p) \neq 0$. Construct a homomorphism

$$\varphi : H_{j-1}(L, Z_p) \to H_j(M, Z_p),$$

analogous to that defined in (^{7}), p. 416, for the generalized fibration α for $p = 2$ and the space of unoriented closed curves with a marked point. φ is an epimorphism, as in (^{7}), whence it follows that $H_{j-1}(L, Z_p) \neq 0$. Since O is homeomorphic to M, from the exact sequence of the pair (L, O) we have

$$0 \to H_{j-1}(L, Z_p) \to H_{j-1}(L \bmod OZ_p),$$

so that the latter group is nontrivial. Now from Theorems 16, 19, 18, 20 (^{8}) there follows the existence on M of a closed extremal of index $\leq j-1 < m$, which contradicts the definition of the number m.

The assertion concerning J_a^L will be obtained by analogous arguments as soon as the applicability to J_a^L of the Hurewicz theorem has been proved. To prove this, let us show that $\pi_0(J_a^L) = \pi_1(J_a^L) = 0$ for $m \geq 2$, and $\pi_0(J_a^L) = 0$ for $m = 1$. If $\pi_0(J_a^L) \neq 0$, then one of the components of J_a^L would not intersect O; the shortest closed curve in this component would have index 0, which is impossible for $m \geq 1$. Let now $m \geq 2$. In view of the one-connectedness of O and the covering-homotopy theorem applied to the fibered space $L \to P$, $\pi_1(J_a^L) = 0$ will follow from the following two propositions (cf. item 4):

I. If $\lambda(h) \geq 2$, $b = J(h)$, $b - d$ is sufficiently small, and W is a sufficiently small neighborhood of h in P, then

$$\pi_1(W \cap J_b^P \bmod W \cap J_d^P) = 0.$$

II. If the segment $[c_1 c_2]$ contains no critical values J, then

$$\pi_1(J_{c_2}^L \bmod J_{c_1}^L) = 0.$$

I, with the aid of Morse deformations ((^{4}), Ch. VIII; (^{8}), Ch. V), is reduced to the proof of the relation

$$\pi_1(E_T^k \bmod S_T^{k-1}) = 0 \quad (k = \lambda(h^n)),$$

where E_T^k and, respectively, S_T^{k-1} are obtained from the k-dimensional ball E^k and its boundary S^{k-1} by identifying points congruent with respect to a certain finite group of transformations T (concerning the latter construction see (^{8}), pp. 16, 42). Let σ be the center of E^k, and let $\beta : E^k \setminus \sigma \to S^{k-1}$ be the radial projection; then for all $\tau \in T$, $\beta\tau = \tau\beta$, according to the construction of T, and our assertion follows from the connectedness of S_T^{k-1}.

II is easily proved with the aid of Morse decreasing deformations (cf. (^{6}), § 4, d).

6. Let g be a simple closed extremal of least index $m \geq 1$, $J(g) = c$. From Morse theory ((^{4}), Ch. III) it easily follows that for any n, $\lambda(h^n) \geq \lambda(h)$; therefore g has the least index also among all (not necessarily simple) closed extremals, and m coincides with the number so denoted in item 5. We perform a foliation with singu-

···with the properties of the sphere S^{m+1}, as in (^{6}), Ch. IV; the fibers in this case are either circles, which can be continuously oriented, or points, regarded as one-point oriented curves. As is clear from the construction in (^{6}), the foliation γ has a secant surface. The base of γ is a certain polyhedron $P^m = P_1^m \cup P_2^{m-1}$, where P_1^m is a manifold diffeomorphic to an open ball; $P_2^{m-1} \subset O$; P^m is an integral cycle $L(S^{m+1}) \bmod O(S^{m+1})$. Each map $F : S^{m+1} \to M$ naturally induces $\Phi_F : L(S^{m+1}), O(S^{m+1}) \to L, O$.

As we shall show, for a sufficiently small neighborhood $U(g)$ and a sufficiently small $\varepsilon > 0$ there exists an $F : S^{m+1} \to M$ such that $\Phi_{F*}(P^m) = z_m$ is an integral cycle L mod O hanging on g.

Find such a number ν that J_c^L can be carried by a non-length-increasing Morse deformation into the manifold M of closed extremal polygons with ν links. J defines on M a function ψ with critical point g of index m. In some neighborhood $U(g) \cap M = V$ there exist smooth surfaces K^m, K^r, homeomorphic to balls, with boundaries $\dot{K}^m, \dot{K}^r$, such that $K^m \subset V,\ K^r \subset V,\ K^m \cap (\psi \geq c) = \dot{K}^r \cap (\psi \leq c) = g,\ g \in (\dot{K}^m \cup \dot{K}^r),\ m + r = \dim M$, and, with suitable orientations, the algebraic index of intersection $K^m \times K^r = 1$ on M. We may assume that $K^m \subset L$, since the construction of the corresponding local secant surface L over P presents no difficulty. For sufficiently small $c - a$, $K^m \subset J_a^L$ and, by item 5, there exists a map $\Phi : \bar{P}_1^m \to J_c^L$ such that $\Phi^{-1}(J =' c)$ is a point $q \in P_1^m$, some neighborhood $W(q)$ is mapped by $\Phi|_{\bar{W}}$ diffeomorphically onto K^m, and $\Phi(\bar{P}_1^m, \dot{P}_1^m)$ is a single point $s \in O$. Put $\Phi(P_2^{m-1}) = s$; in view of the existence of a secant surface of the foliation γ, it is not difficult to construct such an $F : S^{m+1} \to M$ that $\Phi = \Phi_F$, and Φ_F satisfies all the requirements.

We now turn to the proof of Theorem 2; as was noted in item 1, this is also sufficient for the proof of Theorem 1. Let first $m \geq 1$. As Schwarz showed ((8), Theorems 14, 14′), $H^i(\bar{P}(S^{m+1}) \bmod O(S^{m+1}), R) = 0$ for $m+1 \geq 3,\ i \leq m+1$. Considering P^m as an integral cycle $\bar{P}(S^{m+1})$ mod $O(S^{m+1})$, we therefore have $P^m \sim_0 0$ in $\bar{P}(S^{m+1})$ mod $O(S^{m+1})$, where $\sim_0$ denotes weak homology. For $m + 1 = 2$, $2P^m \sim 0$ according to the direct construction in (5), item 3. With the aid of the map $\bar{P}(S^{m+1}) \to \bar{P}$, induced by F, we obtain $z_m \sim_0 0$ in $\bar{P}$ mod O. For $m = 0$ put $z_0 = g$; by connectedness of $\bar{P}$, $z_0 \sim 0$ mod O. We shall show that there exists on M a closed extremal h with type number $m_{\bar{P},R}^{m+1}(h) \neq 0$. Indeed, suppose that there is no such extremal. From the result of item 4 it follows easily that, for arbitrarily small $\varepsilon > 0$, $z_m \sim_0 0$ in $J_{c+\varepsilon}^{\bar{P}}$. Applying Morse deformations, we find from this that, for some integer ν, $\nu K^m \sim 0$ on $(\psi \leq c)$ mod $(\psi < c)$ (see item 6); but this is impossible in view of the intersection $K^m \times K^r = 1$ cited in item 6. Thus the existence of h is proved, and if h is a simple extremal, then, by Theorem 21 of (8), Theorem 2 is also proved.

It remains to consider the case when h is a multiple extremal. Take the corresponding simple extremal h_0; as was already noted in item 6, $\lambda(h_0) \leq \lambda(h)$, so that either $\lambda(h_0) = m$, or $\lambda(h_0) = m + 1$. It turns out that the first case is impossible. Indeed, $\lambda(h_0) = m$ and $m_{\bar{P},R}^{m+1}(h) \neq 0$ are incompatible by item 3. Since $\lambda(h_0) \neq \lambda(g)$, h_0 and g are distinct; by item 5, $m < j$, and the theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.