V. A. Pliss. Nonlocal problems in the theory of oscillations. "Nauka" Publishing House, Moscow, 1964

Authors: V. I. Smirnov

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Abstract

Full Text

Preamble

Differential Equations

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V. A. Pliss. Non-local Problems in the Theory of Oscillations. "Nauka" Publishing House, Moscow, 1964.

This monograph investigates two broad classes of systems of differential equations. The first class consists of systems of the form:

$$\frac{dx}{dt} = P(x, y, z), \quad \frac{dy}{dt} = Q(x, y, z), \quad \frac{dz}{dt} = R(x, y, z)$$

where P, Q, R are periodic functions of the variable z. The second class comprises systems of the form:

$$\frac{dx}{dt} = X(x, t)$$

where $X(x, t)$ is a vector function that is periodic with respect to the time t.

The book is dedicated to the study of the global behavior of solutions for these systems, focusing on problems that arise in the qualitative theory of differential equations and the theory of nonlinear oscillations. The author provides a rigorous treatment of non-local problems, including the existence and stability of periodic solutions, the structure of attractors, and the behavior of trajectories in the phase space.

0L = F (X , t) , (1) dt

where the vector-function depends periodically on time (with period ω), and for systems of the form:

(*X. = F(X). (2) dt

This work primarily examines non-local problems associated with these systems. Among these issues, the most detailed analysis is devoted to the existence of periodic solutions and their global stability. The book consists of three chapters. Chapter I investigates systems of the form (1) of arbitrary order. Specific second- and third-order systems are considered only in certain cases to illustrate the general material. Section 1 is introductory in nature; it provides basic definitions and presents general facts regarding the behavior of solutions for the systems under study. While most of these facts are well-known, some—such as the important theorem on the periodicity of a strongly orbitally stable recurrent motion (Theorem 1.6)—are established here for the first time. The results of Section 1 are extensively utilized throughout the remainder of the work.

A significant portion of the first chapter is dedicated to dissipative systems. Section 2 examines the general properties of dissipative systems, formulating necessary and sufficient conditions for dissipativity in terms of various types of Lyapunov functions. It is proved that a dissipative system possesses the following structure: there exists a bounded, periodic, invariant set S that is stable in the sense of Lyapunov, and all solutions tend toward it as $t \to \infty$. Consequently, all peculiarities of the behavior of solutions for dissipative systems are concentrated within the set S. This set and its cross-section I by the plane $t = 0$ play a crucial role in the subsequent study of dissipative systems. Sections 3-5 formulate various sufficient conditions for the dissipativity of systems (1) and the existence of harmonics within them. Section 3 studies multidimensional systems that are, in various senses, close to linear. Sections 4 and 5 provide sufficient conditions for the dissipativity of certain second- and third-order systems. To prove dissipativity here, various functions are constructed such that their increment along solutions over a period is negative. Combined with the general theorems of Section 2, this allows for the establishment of dissipativity.

Section 6 stands somewhat apart in Chapter I, as it investigates the existence of periodic solutions for systems of arbitrary order that are not necessarily dissipative. The primary role here is played by the existence of a fixed point of the so-called T-transformation, which maps a point in the space at $t = 0$ to a point $(X_0, \omega, 0)$. In some cases, the existence of a fixed point for this transformation follows from Krasnoselskii' s theorem. Based on this, conditions are provided for the existence of periodic solutions for systems that are, in a certain sense, close to linear. These conditions are constructive in nature. The final two sections of Chapter I are devoted to the study of systems with convergence—that is, systems of the form (1) which possess a single globally stable periodic solution.

In other words, a system possessing the property of convergence is a dissipative system where the set S consists of a single periodic solution, or I consists of a single point. General properties of systems with convergence are formulated, and general necessary and sufficient conditions for the presence of convergence are provided. A number of specific systems are studied for which convergence is established.

The first three sections of Chapter II (Sections 9–11) are devoted to the study of the scalar equation of the form (1). Section 9 describes the arrangement of the integral curves of such equations and investigates several questions related to periodic solutions. Particular attention is paid to equations with a polynomial right-hand side—that is, equations where the function is a polynomial in x with coefficients periodic in t. For such equations, the qualitative picture of the integral curves is refined, and the question of the possible number of periodic solutions is studied in detail. Sections 10 and 11 study the scalar equation (1) in which the function depends periodically not only on t but also on x. It is geometrically convenient to consider such equations on a torus. Section 10 presents well-known facts from the Poincaré–Denjoy theory regarding the behavior of trajectories on a torus. These facts are utilized in subsequent sections (Sections 11, 14, 15). Section 11 examines a specific problem: the perturbation of a differential equation with a right-hand side that is periodic in both arguments. Here, the stability of the rotation number on the torus is investigated, and necessary and sufficient conditions are established for equation (1) to be structurally stable (rough) on the torus.

The remaining six sections of Chapter II study second-order systems. Section 12 establishes a series of general theorems on the behavior of solutions for such systems. In particular, it proves Brouwer' s fundamental theorem on transformations of the Euclidean plane into itself. From this theorem, Massera' s theorem is derived regarding the existence of a periodic solution for system (1) in cases where all solutions are extendable over the interval $0 \leq t < \infty$ and there exists a solution bounded for $t \geq 0$. The significance of the first condition is clarified, noting that it is non-essential when $n = 1$. Furthermore, Section 12 provides a detailed study of the properties of a second-order dissipative system. It is shown that the total index of the periodic solutions of such a system is equal to 1. The structure of the boundary of the set S for a two-dimensional dissipative system is examined, and the possibility of periodic solutions appearing on the boundary of S is investigated.

Section 13 considers systems possessing a smooth, torus-like invariant surface Σ that is asymptotically stable in the sense of Lyapunov. Sufficient conditions are provided for "nearby" systems to possess the same invariant surface. Section 14 investigates a similar question but under entirely different assumptions regarding the behavior of the integral curves on the surface Σ. Because the methods of Section 13 are inapplicable in this context, the author employs a technique related to Lyapunov's first method. Section 13 demonstrates that such invariant surfaces can serve as the boundary of the set S for two-dimensional dissipative systems. In this case, the set I has a very simple structure: it represents a region bounded by a closed Jordan curve. However, not every two-dimensional dissipative system possesses a set with such a simple structure. Section 15 studies a specific second-order differential equation of a special type (N. Levinson). This equation represents a dissipative system with a very complex structure for the set S. It is shown that asymptotically stable periodic solutions with different periods are located on the boundary of this set. It follows that the boundary of the set I is not a closed Jordan curve. Section 16 considers systems in which not all solutions are extendable over a full period. In two such cases, the existence of at least one ω-periodic solution is proved.

Systems of the form (1) are studied in Chapter III of the book. Section 18 establishes general theorems on the existence of periodic solutions, their uniqueness, and their global stability. Additionally, considerable space is devoted to the study of specific third-order systems. Section 19 examines a three-dimensional system with a single nonlinearity, where this nonlinearity satisfies generalized Hurwitz conditions in the neighborhood of infinity but fails to satisfy them in the neighborhood of the origin. This allows for the application of the well-known torus principle to prove the existence of periodic solutions. The final three sections of the book are dedicated to the study of a third-order equation with a single nonlinearity that satisfies the generalized Hurwitz condition for all values of the argument. For this equation, several theorems regarding the behavior of solutions are formulated; in particular, it is proved that solutions either tend toward an equilibrium state or the equation possesses a periodic solution. The final section provides sufficient conditions for the existence of a periodic solution.

The material in the book is essentially divided into two parts. The first part (Sections 1, 2, 7, 9, 10, 12, 18) covers various general propositions concerning the systems and establishes theorems valid for entire classes of systems—dissipative, convergent, second-order, etc. The second part comprises the remaining content of the book, which concerns the study of specific systems. This division of the material into two parts is easily discernible from the brief summary of the book's contents provided above. Naturally, the book does not cover all modern methods of the non-local theory of oscillations. We should point out two major sections that were not included in the book but are covered elsewhere.

in other monographs. The first is the construction of a periodic solution using the small parameter method in the presence of an initial periodic solution (the Poincaré method). The corresponding theory is presented with great completeness in the monographs of I. G. Malkin. Furthermore, the book does not consider two-dimensional autonomous systems at all; the theory of such systems is thoroughly detailed in the book by A. A. Andronov, A. A. Vitt, and S. E. Khaikin. The central issue of this monograph is the investigation of systems with a periodic dependence on time t (the first two chapters). In this regard, the monograph is entirely original and unified by a common idea. Approximately half of the book' s content is based on results obtained by the author himself (some of these results were published in the periodical press without proofs, and some have never been published before). I believe that V. A. Pliss's monograph is highly valuable for specialists in nonlinear oscillations and the qualitative theory of ordinary differential equations. I am unaware of any similar works abroad.

V. I. Smirnov, Member of the Academy

Note: Figure translations are in progress. See original paper for figures.