MATHEMATICAL PHYSICS

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Abstract

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ON THE THEORY OF INCOMPLETE MACDONALD FUNCTIONS

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The functions

$$B_m(x,z) = \frac{1}{2}\int_x^\infty \xi^{m-1} e^{-\frac{1}{2}z\left(\xi+\frac{1}{\xi}\right)}\,d\xi, \tag{1a}$$

$$K_m(x,z) = \frac{1}{2}\int_0^x \xi^{m-1} e^{-\frac{1}{2}z\left(\xi+\frac{1}{\xi}\right)}\,d\xi \tag{1 }$$

we call **incomplete Macdonald functions**.

The functions $B_m(x,z)$ and $K_m(x,z)$ occur in the solution of partial differential equations of parabolic type, for example, the stationary equation of turbulent diffusion of a radioactive impurity in the atmosphere with allowance for wind transport (u_z = const) from local sources. Then, when the coefficient of turbulence varies with height ($k_z = k_0 z^n$; $0 \geq n \geq 1$), the concentration $q(x,z) = A(z)K_m(x,z)$.

The function $K_m(x,z)$ also occurs in calculations of the interaction effect of gamma quanta with the scintillator material. In the latter case the recorded signal $g(E,E_0)$, where E is the signal amplitude, has the form of a Gaussian curve with variance proportional to the energy of the incident quanta E_0. If the spectrum $\omega(E_0)$ of the incident radiation is continuous, then the resulting effect is

$$g(E,E_1) = 2\sum_{m=1}^{\infty} a_{m-1}E^{-m}e^{\beta E}K_m(E_1,\beta E).$$

Here a_{m-1} are the coefficients of the Taylor-series expansion of the product $\omega(E_0)\rho(E_0)$, where $\rho(E_0)$ denotes the detector efficiency; β is a constant coefficient depending on the resolving power of the spectrometer.

One could also indicate a number of other applications of incomplete Macdonald functions.

It is easy to show that

$$K_m(x,z) = B_{-m}\left(\frac{1}{x}, z\right). \tag{2}$$

Let us integrate (1) by parts; then we obtain the following recurrence formulas for the functions $B_m(x,z)$ and $K_m(x,z)$:

$$B_{m+1}(x,z) = \frac{2m}{z} B_m(x,z) + B_{m-1}(x,z) + \frac{x^m}{z} e^{-\frac{1}{2}z\left(x+\frac{1}{x}\right)}, \tag{3a}$$

$$K_{m+1}(x,z) = \frac{2m}{z} K_m(x,z) + K_{m-1}(x,z) - \frac{x^m}{z} e^{-\frac{1}{2}z\left(x+\frac{1}{x}\right)}. \tag{3 }$$

Formulas (3) are valid for an arbitrary number m. In the particular case, from formula (3a) as $x \to 0$ and from formula (3) as $x \to \infty$, the known recurrence formula for Macdonald functions is obtained. The same recurrence formula is obtained from the sum of formulas (3a) and (3).

1. Consider the function $B_m(x,z)$ of arbitrary order. We introduce the concept of an integro-exponential function with negative index

$$E_{-m}(x) = \int_0^\infty u^m e^{-xu}\, du. \tag{4}$$

It is easy to show that

$$E_{-m}(x) = \frac{1}{x^{m+1}} \Gamma(m+1, x),$$

where

$$\Gamma(m,x) = \Gamma(m) - \int_0^x u^{m-1} e^u\, du = \Gamma(m) - \gamma(m,x).$$

Here $\Gamma(m)$ is the gamma function; $\gamma(m,x)$ is the tabulated incomplete gamma function [1]. If in (1) we put $\xi = xt$, then

$$B_{m+1}(x,z) = \frac{x^{m+1}}{2} \int_0^\infty t^m e^{-at} \sum_{p=1}^\infty \frac{1}{p!} \left(\frac{b}{t}\right)^p dt \tag{5}$$

$$\left(a = -\frac{1}{2}xz; \quad b = -\frac{1}{2} \cdot \frac{z}{x}\right).$$

Integration of (5) for arbitrary m gives

$$B_{m+1}(x,z) = \frac{x^{m+1}}{2} \sum_{k=0}^{\infty} \frac{1}{k!} b^k E_{k-m}(a), \tag{6}$$

where

$$E_m(x) = \int_1^{\infty} u^{-m} e^{xu}\, du$$

is the tabulated integro-exponential function [2]. For $m < 0$

$$B_{-m}(x,z) = \frac{x^m}{2} \sum_{k=0}^{\infty} \frac{1}{k!} b^k E_{k+m+1}(a). \tag{7}$$

From formulas (2) and (7) one easily obtains the following expression for the function $K_m(x,z)$ for $m > 0$:

$$K_m(x,z) = \frac{x^m}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} a^k E_{k+m+1}(-b). \tag{8}$$

2. For positive integer m, using expression (5), one can obtain the following formula for computing the function $B_m(x,z)$:

$$\begin{aligned} B_{m+1}(x,z) = \frac{x^{m+1}}{2} \Bigg\{ m!\, \frac{e^{-a}}{a} \Bigg[\frac{1}{m!} + \frac{1}{(m-1)!}\left(\frac{1}{a} + \frac{b}{m}\right) + \\ + \frac{1}{(m-2)!}\left(\frac{1}{a^2} + \frac{b}{am} + \frac{b^2}{2!\, m(m-1)}\right) + \cdots \\ \cdots + \frac{1}{2!}\left(\frac{1}{a^{m-2}} + \frac{b}{ma^{m-3}} + \cdots + \frac{b^{m-2}}{(m-2)!\, m!/2}\right) + \\ + \frac{1}{1!}\left(\frac{1}{a^{m-1}} + \frac{b}{ma^{m-2}} + \cdots + \frac{b^{m-1}}{(m-1)!\, m!}\right) + \\ + \left(\frac{1}{a^m} + \frac{b}{ma^{m-1}} + \cdots + \frac{b^m}{(m!)^2}\right)\Bigg] + \sum_{m=1}^{\infty} \frac{t^k}{k!} E_{k-m}(a) \Bigg\}. \end{aligned} \tag{9}$$

The series (6), (7), (8), and (9) converge for any values of x and z, owing to the fact that

$$E_n(y) > E_{n+1}(y).$$

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References

[1] V. I. Pagurova, *Tables of the Incomplete Gamma Function*, Moscow, 1963.
[2] V. I. Pagurova, *Tables of the Integro-Exponential Function*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.