Mathematics

1965

SovietRxiv

Abstract

Full Text

Mathematics

WANG TUN

THEORY OF THE HEAT POTENTIAL

(Presented by Academician I. G. Petrovskii on 21 IX 1964)

1°. The function

$$G(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{(x-\xi)^2}{\psi(t-\tau)}\right\} \tag{1}$$

is the fundamental solution of the heat-conduction equation.

By $g = g_0^T$ we shall denote an open domain bounded by two straight lines $t = 0$, $t = T > 0$ and by two nonintersecting curves $S_i = (S_i)_0^T$ $(i = 1, 2)$, defined by the equations $x = \varphi_i(t)$, $\varphi_1(t) < \varphi_2(t)$, $0 \le t \le T$. In what follows, by S (or φ) we shall mean either $S_1[\varphi_1]$ or $S_2[\varphi_2]$.

Consider in g the heat single-layer potential

$$V(x,t) = \int_{S_0^t} \mu(\tau) G(x,t;\xi,\tau)\, ds_\tau \equiv V[\mu], \tag{2}$$

the heat double-layer potential

$$W(x,t) = \int_{S_0^t} \mu(t) \frac{\partial G(x,t;\xi,\tau)}{\partial \xi}\, ds_\tau \equiv W[\mu] \tag{3}$$

and the planar heat potential

$$U(x,t) = \iint_{g_0^t} \rho(\xi,\tau) G(x,t;\xi,\tau)\, d\xi\, d\tau \equiv U[\rho], \tag{4}$$

where $\mu(t)$, $\rho(x,t)$ are the densities of the corresponding potential.

By the direct value $W_{\text{pr}}(t)$ of the potential W is meant the integral (3) when x varies only along the curve S. The direct value $V_{\text{pr}}(t)$ of the derivative with respect to x of the potential (2) is defined analogously.

Membership of a function $\varphi(t)$ (or $\varphi(x)$) of one variable t (or x) in the class $C^{(n,\lambda)}$ $(n \ge 0,\ 0 < \lambda \le 1)$ is written in the form $\varphi \in (n, \lambda)$. Continuity of the function $\varphi(t)$ is denoted by the symbol $\varphi \in (0, 0)$.

A function $P(x,t)$ of two variables belongs in g to the class (m,n,α,β), $m,n \geq 0$, $0 \leq \alpha,\beta \leq 1$, if it has a continuous derivative of the form $\partial^{m+n}P(x,t)/\partial x^m \partial t^n$ and if the quantity

$$P_{(m,n,\alpha,\beta)} = \sup_{\substack{(x,t)\in g \\ (\bar{x},\bar{t})\in g}} \left\{ \left| \frac{\partial^{m+n}P(x,t)}{\partial x^m \partial t^n} - \frac{\partial^{m+n}P(\bar{x},\bar{t})}{\partial x^m \partial t^n} \right| \Big/ \left(|x-\bar{x}|^\alpha + |t-\bar{t}|^\beta\right) \right\}$$

is finite.

The class $B^{(n,\lambda)}$ is the direct sum of $n+1$ classes of the form (p,q,α,β), in which p,q,α,β are determined by the relations*

$$\begin{gathered} p = 0,1,2,\dots,n; \qquad q = [(n-p)/2]; \\ \alpha = 2\beta = \lambda \quad \text{for } p = n, n-2, n-4, \dots; \\ \alpha = 2\beta - \lambda = 1 \quad \text{for } p = n-1, n-3, n-5, \dots. \end{gathered} \tag{5}$$

The norm of the function $\varphi(t)$ in the class (n,λ) is defined in the usual way (see ([1])). The norm of the function $P(x,t)$ in the classes (m,n,α,β) and $B^{(n,\lambda)}$ is deter–

* The classes $B^{(n,\lambda)}$ $(0 \leq n \leq 2)$ were used in ([2],[3]).

is as follows:

$$\|P\|_{(m,n,\alpha,\beta)} = P_{(m,n,\alpha,\beta)} + \max_{(x,t)\in g} |P(x,t)|,$$

$$\|P\|_{B^{(n,\lambda)}} = \max_{(p,q,\alpha,\beta)} \{\|P\|_{(p,q,\alpha,\beta)}\},$$

where (p,q,α,β) runs through all values defined in (5).

2°. For a given smoothness of the curve S and of the density $\mu(t)$ on $\bar{S}$, the smoothness of the contour heat potentials $W[\mu]$ and $V[\mu]$ in the domain g_δ^T is shown in Table 1, where $\delta > 0$ is arbitrary. In Table 1, $\gamma < 1$ is arbitrary; $\lambda'' = \lambda$ for

Table 1

$\eta(t)$	$\eta(t)$	s	$W[\mu]$	$V[\mu]$
$(0,\lambda)$	$\lambda = 0$	$(0,0)$		$B^{(0,\gamma)}$
$(0,\lambda)$	$0 \leq \lambda \leq \frac{1}{2}$	$(1,0)$	$B^{(0,2\lambda'')}$	$B^{(1,2\lambda'')}$
$(0,\lambda)$	$\frac{1}{2} < \lambda \leq 1$	$(1,\lambda-\frac{1}{2})$	$B^{(1,2\lambda'-1)}$	$B^{(2,2\lambda'-1)}$
$(1,\lambda)$	$0 \leq \lambda \leq \frac{1}{2}$	$(1,\lambda)(2,0)$	$B^{(1,2\lambda'')}B^{(2,2\lambda'')}$	$B^{(2,2\lambda'')}B^{(3,2\lambda'')}$

$\eta(t)$	$\eta(t)$	s	$W[\mu]$	$V[\mu]$
$(1,\lambda)$	$\frac{1}{2} < \lambda \le 1$	$(2,0)\left(2, \lambda - \dfrac{1}{2}\right)$	$B^{(3,0)}\, B^{(3,2\lambda'-1)}$	$B^{(4,0)}\, B^{(4,2\lambda'-1)}$
$(n,\lambda)(n \ge 2) 0 \le \lambda \le 1$	$0 \le \lambda \le \frac{1}{2}$	$(n,\lambda)\left(n, \lambda + \dfrac{1}{2}\right)$	$B^{(2n-1,2\lambda'')}\, B^{(2n,2\lambda'')}$	$B^{(2n,2\lambda'')}\, B^{(2n+1,2\lambda'')}$
$(n,\lambda)(n \ge 2) 0 \le \lambda \le 1$	$\frac{1}{2} < \lambda \le 1$	$(n,\lambda)\left(n+1, \lambda - \dfrac{1}{2}\right)$	$B^{(2n,2\lambda'-1)}\, B^{(2n+1,2\lambda'-1)}$	$B^{(2n+1,2\lambda'-1)}\, B^{(2n+2,2\lambda'-1)}$

$0 \le \lambda < \frac{1}{2}$, and $\lambda'' < \frac{1}{2}$ is arbitrary for $\lambda = \frac{1}{2}$; $\lambda' = \lambda$ for $\frac{1}{2} < \lambda < 1$, and $\lambda' < 1$ is arbitrary for $\lambda = 1$; moreover the norm of $W[\mu]$ ($V[\mu]$) is estimated in terms of the corresponding norm of the density $\mu(t)$; when

$$\mu(0) = \mu'(0) = \cdots = \mu^{(n)}(0) = 0,$$

one may take $\delta = 0,\ n \ge 0$.

We note that Piskorek (4) proved that if the density μ is bounded and integrable, then $V[\mu] \in B^{(0,\gamma)}$ in $\bar{g}$. Gevrey (2) proved the following facts. Let $S \in (0,\sigma)$, $\sigma > \frac{1}{2}$, $\mu \in (0,\lambda)$, and $\mu(0) = 0,\ 0 < \lambda \le 1$; then in $\bar{g}$, $W[\mu] \in (0,\lambda')$ with respect to t, and $W[\mu] \in (0, 2\lambda)$ with respect to x for $\lambda \le \frac{1}{2}$. Let $S \in (1,\lambda)$, $\mu \in (1,\lambda)$, and $\mu'(0) = 0,\ 0 < \lambda \le \frac{1}{2}$; then in $\bar{g}$,

$$\partial W[\mu]/\partial t \in (0,\lambda)$$

with respect to t, and

$$\partial W[\mu]/\partial x \in (0, \lambda'' + \tfrac{1}{2})$$

with respect to t.

3°. For a given smoothness of the curves S_i $(i = 1,2)$ and of the density $\rho(x,t)$ in $\bar{g}$, the smoothness of the plane heat potential $U[\rho]$ in $\bar{g}_\delta^T$ ($\delta > 0$ arbitrary) is shown in Table 2, where $0 < \lambda \le 1$, $0 < \bar{\lambda} < \lambda$ is arbitrary, and $\varepsilon > 0$ is arbitrary; moreover the norm of $U[\rho]$ in the class $B^{(n+2,\lambda)}(\bar{g}_\delta^T)$ is estimated in terms of the norm of the density ρ in the class $B^{(n,\lambda)}(\bar{g})$; when $\rho \in B^{(0,0)}$, one may take $\delta = 0$; $\delta = 0$ may also be taken in the case $\rho \in B^{(n,\lambda)}$, $n \ge 0$, if at the points $(\varphi_i(0), 0)$ $(i = 1,2)$ the density and all its derivatives required by the class $B^{(n,\lambda)}$ are equal to zero.

When $\rho \in B^{(0,0)}$, the result indicated in Table 2 belongs to Gevrey (2). He also proved that if $S \in (0,\sigma)$, $\sigma > \frac{1}{2}$, $\rho \in (0,\lambda)$ with respect to t in $\bar{g}$, $0 < \lambda \le \frac{1}{2}$, then in $\bar{g}$

$$\partial U[\rho]/\partial t \in (0,\lambda)$$

and

$$\partial U[\rho]/\partial x \in (0, \lambda'' + \tfrac{1}{2})$$

with respect to t.

4°. For a known smoothness of the curve S, the direct values $W_{\text{dir}}(t)$ and $V_{\text{dir}}(t)$ increase the smoothness of their density $\mu(t)$ by half a unit. The smoothness of $W_{\text{dir}}(t)$ and $V_{\text{dir}}(t)$ on the curve $\overline{S}_{\delta}^{T}$ is given in Table 3, where $\delta > 0$ is arbitrary; here the norm of W_{dir} (V_{dir}) in the class to which membership is proved is estimated in terms of the norm of the density in the corresponding class;

Table 2

$\rho(xt)$	$s_i,\ i = 1, 2$	$U[\rho]$
$B^{(0,0)}$	$(0, 0)$	$B^{(1,1-\varepsilon)}$
$B^{(0,\lambda)}$	$(1, 0)$	$B^{(2,\bar{\lambda})}$
$B^{(1,\lambda)}$	$(1, \lambda/2)$	$B^{(3,\bar{\lambda})}$
$B^{(2,\lambda)}$	$(2, 0)$	$B^{(4,\bar{\lambda})}$
$B^{(3,\lambda)}$	$(2, \lambda/2)$	$B^{(5,\bar{\lambda})}$
$B^{(n,\lambda)},\ n \geqslant 4$	$(n/2 + 1, \lambda/2),\ n$ even; $((n + 1)/2, (1 + \lambda)/2),\ n$ odd.	$B^{(n+2,\bar{\lambda})}$

when $\mu \in (0, 0)$, one may take $\delta = 0$; $\delta = 0$ may also be taken in the remaining cases if $\mu(0) = \mu'(0) = \cdots = \mu^{(n)}(0) = 0,\ n \geqslant 0$.

Table 3

$\mu(t)$	s	$W_{\text{dir}}(t)\ (V_{\text{dir}}(t))$
$(0, 0)$	$(0, \gamma),\ 3/4 < \gamma \leqslant 1$	$(0,\ 2\lambda - 3/2)$
$(n, \lambda),\ n \geqslant 0$	$(n + 1, \lambda)$, $0 \leqslant \lambda \leqslant 1/2(n + 1, \lambda)$, $1/2 < \lambda \leqslant 1$	$(n, \lambda'' + 1/2)(n + 1, \lambda - 1/2)$

5°. By an inverse problem in the theory of the heat potential we mean a problem in which, starting from the known smoothness of the potential itself, the smoothness of its density is studied. Let $g^s_{\pm\varepsilon}$ be some open neighborhood of the curve S:

$$g^s_{\pm\varepsilon} = \{(x, t),\ \min(\varphi(t) \pm \varepsilon) < x < \max(\varphi(t), \varphi(t) \pm \varepsilon),\ 0 < t < T\},$$

where $\varepsilon > 0$ is an arbitrary number.

For a given smoothness of the curve S and of the potential $W[\mu]$ ($V[\mu]$) in $\bar{g}^s_{\pm\varepsilon}$, the smoothness of the density $\mu(t)$ on the curve $\overline{S}_{\delta}^{T}$ is given in Table 4, where $\delta > 0$ is arbitrary; here the norm of the density $\mu(t)$ is estimated in terms of the corresponding norm of the potential $W(V)$; when $S \in (1, 0)$, one may take $\delta = 0$.

Table 4

$W[\mu]$	$V[\mu]$	s	$\mu(t)$
$(00\ \gamma\gamma),\ 0<\gamma\leqslant 1/2$	$(10\ \gamma\gamma)$	$(1,0)$	$(0,\gamma)$
$\sum\limits_{\substack{p+q=n\\ 0\leqslant p,\ q\leqslant n}} (pq\lambda\lambda)$	$\sum\limits_{\substack{p+q=n\\ 0\leqslant p,\ q\leqslant n}} (p+1\ q\lambda\lambda)$	$(n,\lambda+1/2),\ 0\leqslant\lambda\leqslant 1/2(n+1,\lambda-1/2),\ 1/2<\lambda<1$	(n,λ)

6°. It is easy to see that, using the smoothness properties of heat potentials and the smoothness-improving properties of the direct values of heat potentials, one can obtain boundary estimates in the classes $B^{(n,\lambda)}$ for solutions of boundary-value problems for the heat-conduction equation (see ([5,6])). These estimates

refine the Gilbert-Friedman estimates ([7,8]) and are an analogue of the known Schauder estimates for solutions of boundary value problems for an elliptic equation. Estimates in the class $B^{(2,\lambda)}$ for solutions of the third boundary value problem for a parabolic equation were obtained in ([3]).

We now consider the following boundary value problems for the heat-conduction equation with discontinuous coefficients.

Let $S_j = \{(x,t), x=\varphi_j(t), 0<t<T\}$, $j=1,2,3$, be three nonintersecting curves. Further, let

$$l_i = \{(x,0), x_i=\varphi_i(0)\le x\le x_{i+1}=\varphi_{i+1}(0)\},$$

$$g_i = \{(x,t), \varphi_i(t)<x<\varphi_{i+1}(t), 0<t<T\}, \qquad i=1,2.$$

We seek functions $u_i(x,t)$ satisfying in the domain g_i the equation

$$M_i u_i \equiv a_i \partial^2 u_i/\partial x^2 - \partial u_i/\partial t = f_i(x,t), \qquad i=1,2 \tag{6}$$

($a=\text{const}>0$), and the following conditions:

$$1. \qquad u_i(x,0)\big|_{l_i} = \chi_i(x), \qquad i=1,2; \tag{7}$$

$$2. \qquad \beta_1(t)\frac{\partial u_1}{\partial x}\bigg|_{S_2} - \beta_2(t)\frac{\partial u_2}{\partial x}\bigg|_{S_2} = \xi(t), \qquad 0\le t\le T; \tag{8}$$

$$\sigma_1(t)u_1\big|_{S_2} - \sigma_2(t)u_2\big|_{S_2} = \zeta(t), \qquad 0\le t\le T, \tag{9}$$

$$3. \qquad u_i=\psi_i(t) \text{ on } S_j \quad \text{(for the first boundary value problem)} \tag{10}$$

or

$$\partial u_i/\partial x + \alpha_i(t)u_i = \psi_i(t) \text{ on } S_j \quad \text{(for the third boundary value problem)}, \tag{11}$$

where $j = 1$ for $i = 1$ and $j = 3$ for $i = 2$, $0 \le t \le T$.

Let

$$\sqrt{a_1}\beta_1(t)\sigma_2(t) + \sqrt{a_2}\beta_2(t)\sigma_1(t) \neq 0$$

for $0 \le t \le T$, and suppose that problems I (6–10) and II (6–9) are compatible in the class $B^{(m,\lambda)}$ ($m = 1$ or $n+2$, $0 < \lambda \le 1$), i.e., at the points $(x_j, 0)$ ($j = 1, 2, 3$) all the necessary compatibility conditions between f_i, χ_i, ψ_i, ξ, and ζ are fulfilled when $u_i(x,t)$ are considered in the class $B^{(m,\lambda)}(\bar{g}_i)$. Then, for prescribed smoothness of $S_j, f_i, \chi_i, \psi_i, \alpha_i, \beta_i, \sigma_i, \xi$, and ζ, the smoothness of the unique solution $u_i(x,t)$ of problems I and II in $\bar{g}_i$ is shown in Table 5, where $0 < \bar{\lambda} < \lambda$ is arbitrary; moreover, the norm of u_i in the class $B^{(m,\bar{\lambda})}$

Table 5

f_i	χ_i	s_j	ξ and β_i	ζ and σ_i	ψ_i and α_i	u_i
$(0,0,\gamma,0)0 < \gamma < 1$	$(1,\lambda)$	$(1,0)$	$(0,\lambda/2)$	$\left(0, \dfrac{1+\lambda}{2}\right)$	$0, (1+\lambda)/2$, problem I$(0,\lambda/2)$, problem II	$B^{(1,\bar{\lambda})}$
$B^{(n,\lambda)} n \ge 0, 0 < \lambda \le 1$	$(n+2,\lambda)$	n even	$(n/2+1;\lambda/2)$	$(n/2,(1+\lambda)/2)$	$(n/2+1,\lambda/2)$	$(n/2+1,\lambda/2)$, problem I$(n/2,(1+\lambda)/2)$, problem II
$B^{(n,\lambda)} n \ge 0, 0 < \lambda \le 1$	$(n+2,\lambda)$	n odd	$(n+1/2, 1+\lambda/2)$	$((n+1)/2,\lambda/2)$	$((n+1)/2,(1+\lambda)/2)$	$((n+1)/2,(1+\lambda)/2)$, problem I$((n+1)/2,\lambda/2)$, problem II

$$(1 \le m \le n+2)$$

is estimated in terms of the norms of f_j, χ_i, ψ_i, ξ, and ζ in the corresponding classes. The uniqueness and existence of a classical solution of problems I and II were considered in ([9,10]).

The author expresses gratitude to V. A. Il' in for guidance of the work and useful advice, and to A. N. Tikhonov for discussion of the results.

Moscow State University
named after M. V. Lomonosov

Received
20 IV 1964

References

1. K. Miranda, *Equations with partial derivatives of elliptic type*, IL, 1957.
2. M. Gevrey, J. Math. Pure et Appl., 9, No. 1—4 (1913).
3. I. I. Kamynin, V. N. Maslennikova, DAN, 153, No. 3 (1963).
4. Piskorek, Ann. Polon. Math., 8, No. 2 (1960).
5. Van Ty, DAN, 152, No. 6 (1963).
6. N. M. Günter, *The Theory of Potential and Its Application to the Fundamental Problems of Mathematical Physics*, 1953.
7. C. Ciliberto, Ricerche Mat., 3, No. 1 (1954).
8. A. Friedman, J. Math. and Mech., 7, No. 5 (1958).
9. I. I. Kamynin, Sibirskii matematicheskii zhurnal, 4, No. 5 (1963).
10. A. A. Samarskii, DAN, 121, No. 2 (1958).

Note: Figure translations are in progress. See original paper for figures.