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Abstract

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GEOPHYSICS

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QUASI-GEOSTROPHIC INSTABILITY OF OCEAN CURRENTS

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Using the approximate methods developed in meteorological works by Go ([1], [2]) and Thompson ([3]), the stability of geostrophic currents in the ocean is investigated with partial allowance for frictional forces.

If the basic current is described by the equations of the dynamical method

$$fU = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}, \qquad g\rho_0 = -\frac{\partial P}{\partial z}, \qquad \rho_0 = \rho_0(y, z), \tag{1}$$

then the linearized equations of the perturbed motion may be written in the form

$$Lu' + v'U_y + w'U_z - fv' = -\left(\frac{p'}{\rho_0}\right)_x + \nu_1 u'_{zz} + \nu\nabla^2 u'; \tag{2}$$

$$Lv' \qquad\qquad + fu' = -\left(\frac{p'}{\rho_0}\right)_y + \nu_1 v'_{zz} + \nu\nabla^2 v'; \tag{3}$$

$$g\frac{\rho'}{\rho_0} = -\left(\frac{p'}{\rho_0}\right)_z; \tag{4}$$

$$L\rho' + \frac{\partial\rho_0}{\partial y}v' + \frac{\partial\rho_0}{\partial z}w' = 0; \tag{5}$$

$$u'_x + v'_y + w'_z = 0. \tag{6}$$

Here $L = \partial/\partial t + U\partial/\partial x$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$; p' is pressure; ρ' is the density perturbation; g is the acceleration due to gravity; f is the Coriolis parameter; u', v', w' are the components of the velocity perturbations of the basic current U along the axes x, y, z, directed eastward, northward, and upward. Subscripts indicate differentiation with respect to the corresponding variable. Capital letters denote quantities characterizing the basic (unperturbed) state; ρ_0 is the density in the unperturbed state; ν_1, ν are the coefficients of vertical and horizontal turbulent viscosity. Equations (2) and (3) give the vorticity equation $\xi = v'_x - u'_y$:

$$Lv' + Z'_{yv} + Z\,\mathrm{div}(u', v') = \nu_1 v'_{zz} + \nu\nabla^2 v'. \tag{7}$$

In this equation the small terms $U_z w_y$ and $U_{zy} w$ have been omitted, and the notation

$$Z = f - U_y, \qquad Z_y = \beta - U_{yy}, \qquad \beta = df/dy. \tag{8}$$

has been introduced.

We now introduce the assumption of quasi-geostrophy, necessary for filtering out internal gravity waves. To this end, let us assume that in the vorticity equation (7) and in the isopycnicity (adiabaticity) equation (5) the velocity perturbations satisfy the conditions:

$$f\rho_0 v' = p'_x, \qquad f\rho_0 u' = -p'_y. \tag{9}$$

Assuming further that the perturbations have the form of elementary waves

$$u' = u(z)\exp\{ik(x - ct)\} \tag{10}$$

and do not depend on the coordinate y, transverse with respect to the basic current (the same relations hold also for v', w', ρ', p'), we obtain

equations (5) and (8) to the form

$$\sigma w = f[(U - c)v_z - U_{zv}], \qquad \sigma = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}; \tag{11}$$

$$(U - c)v - \left(\frac{\beta}{k^2} + i\nu k\right)v + \frac{i\nu_1}{k}v_{zz} = -\frac{f}{k^2}w_z. \tag{12}$$

The solution of the system (11), (12) under specified boundary conditions in principle makes it possible to determine the eigenvalues c as functions of the wave number k. If these eigenvalues are complex, $c = c_r + ic_i$ and $c_i > 0$, then instability occurs, and the amplitude of the wave disturbances grows exponentially as they propagate.

In order to avoid solving the eigenvalue problem, which involves serious mathematical difficulties, we use an approximate approach and express the characteristic stability parameters in terms of averaged elements of the basic motion. Differentiating (12) with respect to z and substituting the result into (11), we obtain a relation connecting w and v for a given basic current $U(z)$. This relation, together with the initial vorticity equation (12) (in which $v_1 = 0$, since vertical viscosity is significant only in the relatively thin Ekman friction layer), forms the system

$$w_{zz} + \frac{k^2\sigma}{f^2} w + \frac{2k^2}{f} U_{zv} - \frac{k^2}{f}\left(\frac{\beta}{k^2} + i\nu k\right) v_z = 0; \tag{13}$$

$$(U - c)v - \left(\frac{\beta}{k^2} + i\nu k\right) v + \frac{f}{k^2} w_z = 0. \tag{14}$$

For an approximate determination of the eigenvalues c, we pass from the differential equations (13), (14) to finite-difference relations. Placing the origin of coordinates at the zero surface $z = 0$ and assuming that within the baroclinic layer the velocity of the basic current varies linearly with depth, $U(z) = U_{zz}$ ($U_z = \text{const}$, $\sigma \approx \text{const}$), we divide the entire thickness of the baroclinic layer H into four layers by the points $z = 0,\ {}^1/_4 H,\ {}^1/_2 H = \Delta H,\ {}^3/_4 H,\ H$. Functions computed at each discrete point will be assigned below the corresponding index $(0, 1, 2, 3, 4)$. Taking into account the boundary condition $w(0) = w(H)$, needed to exclude forced oscillations, we write equation (14) in finite differences for the levels ${}^1/_4 H$ and ${}^3/_4 H$ (1 and 3):

$$(U_3 - c)v_3 - \left(\frac{\beta}{k^2} + i\nu k\right) v_3 - \frac{f}{k^2}\frac{w_2}{\Delta H} = 0,$$

$$(U_1 - c)v_1 - \left(\frac{\beta}{k^2} + i\nu k\right) v_1 + \frac{f}{k^2}\frac{w_2}{\Delta H} = 0. \tag{15}$$

Forming the sum and difference of these equations, and introducing the notation

$$\overline{v} = \frac{v_3 + v_1}{2}, \qquad \Delta v = \frac{v_3 - v_1}{2}, \qquad \Delta U = \frac{U_3 - U_1}{2}, \qquad \overline{U} = \frac{U_3 + U_1}{2} \tag{16}$$

and writing equation (13) for the level ${}^1/_2 H$, we obtain the system

$$(\overline{U} - c)\overline{v} - \left(\frac{\beta}{k^2} + i\nu k\right) \overline{v} + \Delta U \Delta v = 0,$$

$$(\overline{U} - c)\Delta v - \left(\frac{\beta}{k^2} + i\nu k\right) \Delta v + \Delta U \overline{v} - \frac{f}{k^2 \Delta H} w_2 = 0, \tag{17}$$

$$\frac{k^2\sigma(\Delta H)^2 - 2f^2}{\Delta H f} w_2 + 4k^2 \Delta U \overline{v} - 2k^2 \left(\frac{\beta}{k^2} + i\nu k \right) \Delta v = 0.$$

In deriving this system, the obvious identities have been used

$$(U_3 v_3 + U_1 v_1) = 2(\overline{U}\,\overline{v} + \Delta U \Delta v); \qquad (U_3 v_3 - U_1 v_1) = 2(\Delta U\,\overline{v} + \overline{U}\Delta v);$$

$$U_z = 2\Delta U / \Delta H, \tag{18}$$

and it has also been assumed that $v_2 \approx (v_3 + v_1)/2 = \overline{v}$. Equating to zero the determinant of the system (17), after some transformations we arrive at

to the algebraic equation:

$$(c - \overline{U})^2 + \left(\frac{\beta}{k^2} + i\nu k \right) \frac{1 + 2\alpha}{1 + \alpha} (c - \overline{U}) + \left(\frac{\beta}{k^2} + i\nu k \right)^2 \frac{\alpha}{1 + \alpha} + (\Delta U)^2 \frac{1 - \alpha}{1 + \alpha} = 0. \tag{19}$$

Here the notation $\alpha = -k^2 \sigma H^2 / 8f^2 > 0$ has been introduced, since for stable stratification $\sigma = \dfrac{g}{\rho_0} \dfrac{d\rho_0}{dz} < 0$. Also, for convenience expressing ΔU through the velocity of the current at the ocean surface $U_m = U_{zH}$, by the formula $\Delta U = \frac{1}{4} U_m$, we write the solution of equation (19) in the form

$$c = \overline{U} - \frac{1}{2} \left(\frac{\beta}{k^2} + i\nu k \right) \frac{1 + 2\alpha}{1 + \alpha} \pm$$

$$\pm\, i \frac{U_m}{4(1 + \alpha)} \sqrt{\left(1 + \frac{4\nu^2 k^2}{U_m^2} - \frac{4\beta^2}{U_m^2 k^4} - \alpha^2 \right) - i \frac{8\beta\nu}{U_m^2 k}}. \tag{20}$$

Denoting the real and imaginary parts of the expression under the radical by a and ib, we obtain

$$c_i = -\frac{\nu k}{2} \frac{1 + 2\alpha}{1 + \alpha} \pm \mathrm{Re} \left\{ \frac{U_m}{4(1 + \alpha)} \sqrt{a - ib} \right\}, \tag{21}$$

$$c_r = \overline{U} - \frac{\beta}{2k^2} \frac{1 + 2\alpha}{1 + \alpha} \pm \mathrm{Im} \left\{ \frac{U_m}{4(1 + \alpha)} \sqrt{a - ib} \right\}. \tag{22}$$

One of the two solutions (21), giving $c_i > 0$, leads to instability. After simple transformations of expression (21), the equation of the neutral curve $c_i = 0$, bounding the region of instability, takes the form

Fig. 1

Figure 1: Fig. 1

$$\left[1-\alpha^2-\frac{4}{U_m^2}\left(\frac{\beta^2}{k^4}-\nu^2k^2\right)\right]-\frac{4\nu^2k^2}{U_m^2}(1+2\alpha)^2=0. \tag{23}$$

Fig. 1

For definiteness, in the numerical calculations by formulas (20)–(23) it was assumed that $f = 0.87 \cdot 10^{-4}$ sec^{-1} (which corresponds to a latitude of about 40°), and the depth of the zero surface H was taken equal to 1500 m. The stratification parameter $\sigma = 1.3 \cdot 10^{-5}$ sec^{-2}, which roughly corresponds to the mean density gradient in a 1500 m layer in temperate latitudes. In addition, $\nu = 1 \cdot 10^7$ cm^2/sec, $\beta = 1 \cdot 10^{-13}$ cm$^{-1} \cdot$ sec^{-1}. Figure 1 shows the corresponding stability diagram in the plane U_m, $\lambda = 2\pi/k$. The heavy line is the neutral curve $c_i = 0$; the thin lines are isolines $c_i =$ const (in centimeters per second) inside the instability region. The calculations are somewhat simplified if one uses tables of roots of complex numbers [4]. Comparison of Fig. 1 with diagrams calculated for the cases of absence of viscosity and stratification (these diagrams are not given here) shows that the forces of viscosity stabilize the motion and reduce the region of instability. The role of stratification (and vertical motions) reduces to limiting the instability region on the side of short waves, since in the unstratified case sufficiently short waves become unstable for an arbitrarily small vertical gradient of the velocity of the basic current. As an example,

Let us estimate the parameters of the most unstable disturbances for the Gulf Stream below Cape Hatteras ("separation point"). Let us note that the parameter values chosen by us in constructing the stability diagram correspond precisely to this region. Take the mean surface velocity across the Gulf Stream to be $U_m = 100$ cm/sec. For this value of U_m, from the diagram in Fig. 1 it is easy to calculate $\max|\omega_i| = \max|c_i k| \approx 3.84 \cdot 10^{-6}$ sec^{-1}, which corresponds to $c_i \approx 16.5$ cm/sec and to a wavelength of the most unstable disturbance equal to approximately 280 km. The amplitude of the most unstable wave at this value doubles in the time $t = \ln 2/\omega_i \approx 2.1$ days; the phase velocity, calculated by formula (22), is $c_r = 49$ cm/sec. The absence of special observations does not permit a detailed comparison with the actual picture; however, the few synchronous surveys of nonstationary meanders in the Gulf Stream (5) indicate that the example given yields realistic orders of magnitude. As is seen from Fig. 1, for the values of stratification and viscosity chosen by us the critical value of U_m is approximately 8 cm/sec, which for $H = 1500$ m corresponds to a velocity gradient of the basic current $U_z = 5.33 \cdot 10^{-5}$ sec^{-1}. The corresponding critical wavelength is $\lambda_c = 330$ km. Consequently, many systems of ocean currents, such as, for example, the Gulf Stream, Kuroshio, and the North Equatorial Current, are beyond the stability boundary and cannot have a stationary character even at great depths (of the order of 1000 m), which, in particular, is confirmed by observations made with buoy stations.

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Note: Figure translations are in progress. See original paper for figures.