A NOTE ON ORDERS WITH CYCLIC INDEX

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Abstract

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1°. Let $\mathfrak{o}$ be a Dedekind ring, k its field of fractions, K/k a finite separable extension, and $\mathfrak{D}$ the integral closure of the ring $\mathfrak{o}$ in the field K. By a full $\mathfrak{o}$-module in K one means a finitely generated $\mathfrak{o}$-submodule of the field K containing $(K : k)$ elements linearly independent over k. A full $\mathfrak{o}$-module in K containing the identity element of the field and being a ring is called an $\mathfrak{o}$-order of the field K. Every $\mathfrak{o}$-order Λ of the field K is contained, as is known, in the maximal $\mathfrak{o}$-order $\mathfrak{D}$.

In the paper ([1]) it is shown that if the $\mathfrak{o}$-order Λ has cyclic index in $\mathfrak{D}$ (see the definition below), then every finitely generated torsion-free Λ-module decomposes into a direct sum of Λ-modules that are Λ-isomorphic to ideals of the ring Λ. On the other hand, in Bass' s paper ([2]) it is established that if R is a Noetherian integral domain whose integral closure (in its field of fractions) is a finitely generated R-module, then every finitely generated torsion-free R-module decomposes into a direct sum of R-ideals if and only if every ideal of the ring R is generated by no more than two elements. (The integral closure of such a ring R is a Dedekind ring.)

It turns out that the following theorem is true (in the notation and assumptions given above).

Theorem. *In order that every finitely generated torsion-free Λ-module decompose into a direct sum of Λ-ideals, it is necessary and sufficient that the $\mathfrak{o}$-order Λ have cyclic index in the maximal $\mathfrak{o}$-order $\mathfrak{D}$ of the field K.*

The sufficiency of the condition, as already noted, was proved in ([1]). The necessity follows from the result of Bass ([2]) cited above and Lemma 1 below.

2°. Definition. Let $\mathfrak{D}$ be a commutative ring with identity and Λ its subring whose identity coincides with the identity of the ring. We say that Λ has cyclic index in $\mathfrak{D}$ if the factor-module $\mathfrak{D}/\Lambda$, as a Λ-module, is cyclic (i.e., generated by one element).

The cyclicity of the index of Λ in $\mathfrak{D}$ is evidently equivalent to the existence of an element $\omega \in \mathfrak{D}$ such that $\mathfrak{D} = \Lambda + \Lambda\omega$.

Lemma 1. *If the ring $\mathfrak{D}$, as a Λ-module, admits a system of Λ-generators consisting of two elements, then Λ has cyclic index in $\mathfrak{D}$.*

The proof of Lemma 1 is based on the following assertion.

Lemma 2. *Suppose that the ring $\mathfrak{D}$ is a finitely generated Λ-module. If elements $\lambda_1, \dots, \lambda_m$ of Λ are relatively prime in $\mathfrak{D}$ (i.e., generate the unit ideal in $\mathfrak{D}$), then they are relatively prime also in the ring Λ.*

Proof. Let $\mathfrak{D} = \Lambda\omega_1 + \dots + \Lambda\omega_n$ and let $\mathfrak{a} = \Lambda\lambda_1 + \dots + \Lambda\lambda_m$. Since, by hypothesis, $\mathfrak{a}\mathfrak{D} = \mathfrak{D}$, we have

$$\omega_i = \sum_{j=1}^{n} \alpha_{ij}\omega_j \quad (1 \le i \le n), \tag{1}$$

where the coefficients α_{ij} belong to the ideal $\mathfrak{a}$. Denote by Δ the deter-

divisor of the matrix $(a_{ij}) - E$, where E is the identity matrix of order n. From (1) it follows easily that $\Delta\omega_i = 0$ for all $i = 1, \dots, n$, and hence $\Delta = 0$. But, since $a_{ij} \in \mathfrak{a}$, we have $\Delta \equiv 1 \pmod{\mathfrak{a}}$. Thus $1 \in \mathfrak{a}$, and Lemma 2 is proved.

Proof of Lemma 1. Let $\mathfrak{D} = \Lambda\omega_1 + \Lambda\omega_2$. Then $1 = \lambda_1\omega_1 + \lambda_2\omega_2$, where $\lambda_1, \lambda_2 \in \Lambda$. By Lemma 2, in the ring Λ there exist elements λ and μ such that $1 = \lambda_1\mu - \lambda_2\lambda$. Put $\omega = \lambda\omega_1 + \mu\omega_2$. Since the systems ω_1, ω_2 and $1, \omega$ are related by a unimodular transformation, it follows that $\mathfrak{D} = \Lambda + \Lambda\omega$, as was required to prove.

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References

[1] Z. I. Borevich, D. K. Faddeev, *Tr. Mat. Inst. im. V. A. Steklova AN SSSR*, **80**, 51 (1965). [2] H. Bass, *Trans. Am. Math. Soc.*, **102**, No. 2, 319 (1962).

Note: Figure translations are in progress. See original paper for figures.