Velocity Field Due to Singularities in a Thin Layer of Variable Thickness

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Velocity Field Due to Singularities in a Thin Layer of Variable Thickness

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In the present note the velocity field in a thin layer of variable thickness due to singularities is considered, as applied to the problem of flow past a cascade of profiles of a radial-axial hydraulic turbine.

I. Consider, in the x, y plane, a thin layer of thickness $h = h(y)$ satisfying the following conditions: $d \ln h/dy$ is continuous and does not change its sign; $\lim\limits_{y\to\infty} h = h_\infty$ and $\lim\limits_{y\to-\infty} h = h_{(-\infty)}$, where $0 < h_\infty < h_{(-\infty)} < \infty$. Place along the x-axis an infinite row of sources $Qh_{(0)}$ ($h_{(0)}$ is the layer thickness at $y = 0$) with period t. If changes of all quantities with respect to z at each point (x, y) are neglected and one sets $\partial\Phi_p/\partial z = 0$, where Φ_p is the velocity potential, then the differential equation for the potential due to such a row of sources (^{1}) is:

$$\frac{\partial^2\Phi_p}{\partial x^2} + \frac{d\ln h}{dy}\frac{\partial\Phi_p}{\partial y} + \frac{\partial^2\Phi_p}{\partial y^2} = Q\delta(y)\sum_{n=-\infty}^{\infty}\delta(x - nt), \tag{1}$$

where $\delta(y)$, $\delta(x - nt)$ are Dirac delta functions.

In paper (^{1}) it is proposed to seek the solution of equation (1) in the form of a Fourier series, whose coefficients are determined from a system of ordinary differential equations with boundary conditions

$$\left[h\frac{\partial\Phi_p}{\partial y}\right]_{y=\infty} = -\left[h\frac{\partial\Phi_p}{\partial y}\right]_{y=-\infty} = \frac{1}{2t}Qh_{(0)}. \tag{2}$$

We shall show that the solution obtained with boundary conditions (2) will contain, in addition to the flow from the row of sources, an admissible flow

$$v_p = Qh_{(0)}[h_{(-\infty)} - h_{(\infty)}]/2ht[h_{(-\infty)} + h_{(\infty)}],$$

i.e., the boundary conditions for the solution of equation (1) are the following:

$$[\partial\Phi_p/\partial y]_{y=-\infty} = -[\partial\Phi_p/\partial y]_{y=\infty} = Qh_{(0)}/t[h_{(\infty)} + h_{(-\infty)}]. \tag{3}$$

II. Write the equation for the potential of a concentrated source located at the point $(0,0)$:

$$\frac{\partial^2\Phi_i}{\partial x^2} + \frac{d\ln h}{dy}\frac{\partial\Phi_i}{\partial y} + \frac{\partial^2\Phi_i}{\partial y^2} = Q\delta(x)\delta(y). \tag{4}$$

Represent $\Phi_i = \Phi_{i0} + \varphi_i' + \varphi_i''$, where Φ_{i0} is the potential of the plane flow and $\varphi_i = \varphi_i' + \varphi_i''$ is the potential of the additional flow caused by the variability of the layer thickness, with

$$\frac{\partial^2\Phi_{i0}}{\partial x^2} + \frac{\partial^2\Phi_{i0}}{\partial y^2} = Q\delta(x)\delta(y); \qquad \frac{\partial^2\varphi_i'}{\partial x^2} + \frac{\partial^2\varphi_i'}{\partial y^2} = -\frac{d\ln h}{dy}\frac{\partial\Phi_{i0}}{\partial y};$$

$$\frac{\partial^2\varphi_i''}{\partial x^2} + \frac{\partial^2\varphi_i''}{\partial y^2} = -\frac{d\ln h}{dy}\frac{\partial\varphi_i}{\partial y}, \tag{5}$$

under the boundary conditions

$$\operatorname{grad}\varphi_i'\big|_{r=\infty} = 0; \qquad \operatorname{grad}\varphi_i''\big|_{r=\infty} = 0,$$

where $r = \sqrt{x^2+y^2}$, since $h_{(\pm\infty)} \neq 0$.

Using the Green’ s function, let us write the solution of the equation for $\varphi_{\rm II}'$, satisfying the prescribed boundary conditions:

$$\varphi_{\rm II}' = -\frac{Q}{8\pi^2}\iint_{[-\infty,\infty;-\infty,\infty]} \frac{d\ln h(\eta)}{d\eta}\frac{\eta}{\xi^2+\eta^2}\ln\{(\xi-x)^2+(\eta-y)^2\}\,d\xi\,d\eta. \tag{6}$$

It is easy to show that

$$\frac{\partial\varphi_{\rm II}'}{\partial y} = -\frac{Q}{8\pi}\frac{d\ln h}{dy}\ln\left\{\frac{x^2+y^2}{x^2+(y+a)^2}\right\} + f(x,y), \tag{7}$$

where $f(x,y)$ is bounded everywhere and $f(x,y)\to 0$ as $r\to\infty$.

Thus, the solution of equation (5) can be written in the form of the integral equation

$$\varphi_{\rm II}''(x,y) = -\frac{1}{4\pi}\iint_{[-\infty,\infty;-\infty,\infty]} \frac{d\ln h(\eta)}{d\eta}\frac{\partial\varphi_{\rm II}(\xi,\eta)}{\partial\eta}\ln\{(\xi-x)^2+(\eta-y)^2\}\,d\xi\,d\eta. \tag{8}$$

We shall show, by the method of iterations, that for layers with $h(-\infty)/h(\infty) < e^2$ there exists a solution of equation (8), bounded in the whole plane and tending to zero as $r \to \infty$.

Successive approximations are defined as follows:

$$\frac{\partial\varphi''_{\mathrm{II}i}(x,y)}{\partial y} = \frac{1}{2\pi}\iint_{[-\infty,\infty;-\infty,\infty]} \frac{d\ln h(\eta)}{d\eta}\left\{\frac{\partial\varphi'_{\mathrm{II}}(\xi,\eta)}{\partial\eta} + \frac{\partial\varphi''_{\mathrm{II}(i-1)}(\xi,\eta)}{\partial\eta}\right\}\frac{(\eta-y)}{(\xi-x)^2+(\eta-y)^2}\,d\xi\,d\eta, \tag{9}$$

where $\partial\varphi''_{\mathrm{II}0}/\partial y = 0$. From equations (7) and (9) it follows that $\partial\varphi''_{\mathrm{II}i}/\partial y$ are bounded everywhere and

$$\lim_{r\to\infty}\frac{\partial\varphi''_{\mathrm{II}i}}{\partial y} = 0,$$

since

$$\lim_{r\to\infty}\frac{\partial\varphi'_{\mathrm{II}}}{\partial y} = 0.$$

We shall prove the convergence of the iterative process. Write equation (8) in operator form $t = A(t)$, where $t = \partial\varphi''_{\mathrm{II}}/\partial y$. Consider the metric space of all approximations t_i $(i = 1, 2, \dots, \infty)$ with norm $\rho(t_j, t_k) = \max|t_j - t_k|$. Then

$$\rho\{A(t_j), A(t_k)\} =$$

$$= \max\left|\frac{1}{2\pi}\iint_{[-\infty,\infty;-\infty,\infty]}\{t_j - t_k\}\frac{d\ln h(\eta)}{d\eta}\frac{(\eta-y)}{(\xi-x)^2+(\eta-y)^2}\,d\xi\,d\eta\right|,$$

i.e.,

$$\rho\{A(t_j), A(t_k)\} \le \frac{1}{2}\rho(t_j,t_k)\left|\int_{-\infty}^{\infty}\frac{d\ln h(\eta)}{d\eta}\,d\eta\right|,$$

since

$$\int_{-\infty}^{\infty}\frac{(\eta-y)}{(\xi-x)^2+(\eta-y)^2}\,d\xi \le \pi,$$

or, finally,

$$\rho\{A(t_j), A(t_k)\} \le \frac{1}{2}\ln\frac{h(-\infty)}{h(\infty)}\rho(t_j, t_k).$$

By the contraction mapping principle [2], the iterative process converges if $h(-\infty)/h(\infty) < e^2$. Hence the solution of (8) is a solution of (5) with the boundary conditions

$$\operatorname{grad}\varphi''_{\mathrm{II}}|_{r\to\infty} = 0,$$

since

$$\lim\frac{\partial\varphi''_{\mathrm{II}}}{\partial x} = 0.$$

III. Now let us obtain the boundary conditions for the flow due to an infinite row of sources corresponding to (1), by summing the velocities from -

of concentrated sources of this series and finding the limit of the sum as $y \to \pm\infty$. Let us write the partial sum

$$s_m = \sum_{n=-m}^{m}\frac{\partial\varphi_{\mathrm{i}n}}{\partial y} = \frac{1}{2\pi}\sum_{n=-m}^{m}\iint_{[-\infty,\infty;\,-\infty,\infty]}\frac{d\ln h(\eta)}{d\eta}\,\frac{\partial\Phi_{\mathrm{i}}(\xi-nt,\eta)}{\partial\eta}\times$$

$$\times\frac{(\eta-y)}{(\xi-x)^2+(\eta-y)^2}\,d\xi\,d\eta$$

or, making the change of variables $\xi_n = \xi - nt$ and denoting ξ_n by ζ,

$$s_m = \frac{1}{2\pi}\left[\sum_{n=-m}^{m}\iint_{[-M,M;\,-N,N]}\frac{d\ln h(\eta)}{d\eta}\,\frac{\partial\Phi_{\mathrm{i}}(\zeta,\eta)}{\partial\eta}\times\right.$$

$$\left.\times\frac{(\eta-y)}{(\zeta+nt-x)^2+(\eta-y)^2}\,d\zeta\,d\eta + R\right];$$

since $\lim_{r\to\infty}\partial\Phi_{\mathrm{i}}/\partial\eta = 0$, it can be shown that $R = R(x,y,m,M,N) < \varepsilon(M,N)$, where $\varepsilon(M,N) \to 0$ as $M \to \infty$ and $N \to \infty$, uniformly with respect to x, y and m for $|x| \le t/2$, $N < |y| < \infty$ and $0 < m < \infty$. Letting m tend to infinity, and using the theorem on integration of a uniformly convergent series, we obtain

$$\frac{\partial\varphi_{\mathrm{p}}}{\partial y} = \frac{1}{2\pi}\left[\frac{\pi}{t}\int_{-N}^{N}d\eta\int_{-M}^{M}\frac{d\ln h(\eta)}{d\eta}\,\frac{\partial\Phi_{\mathrm{i}}(\xi,\eta)}{\partial\eta}\,\frac{\operatorname{sh}\frac{2\pi}{t}(\eta-y)}{\operatorname{ch}\frac{2\pi}{t}(\eta-y)-\cos\frac{2\pi}{t}(\xi-x)}\,d\xi + R\right],$$

or, finally,

$$\lim_{y\to\infty}\frac{\partial\varphi_{\mathrm{p}}}{\partial y}=-\lim_{y\to-\infty}\frac{\partial\varphi_{\mathrm{p}}}{\partial y}=-\frac{1}{2t}\iint_{[-\infty,\infty;\,-\infty,\infty]}\frac{d\ln h(\eta)}{d\eta}\,\frac{\partial\Phi_{\mathrm{i}}(\xi,\eta)}{\partial\eta}\,d\xi\,d\eta.$$

Now, taking into account that

$$\left[\frac{\partial\Phi_{\mathrm{p}0}}{\partial y}\right]_{y=\infty}=-\left[\frac{\partial\Phi_{\mathrm{p}0}}{\partial y}\right]_{y=-\infty},$$

we obtain conditions (3).

IV. In order to obtain a solution of equation (1) in closed form, we apply the Fourier transform with infinite limits to the equation:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-\frac{d\ln h}{dy}\left[\frac{\partial\Phi_0}{\partial y}+\frac{\partial\varphi}{\partial y}\right], \tag{10}$$

where $\Phi=\Phi_0+\varphi$ is the potential of a source and a sink located at the points $(0,0)$ and $(x_0,0)$. The introduction of the sink is due to the fact that the Fourier transform with infinite limits cannot be applied to a single source, since its potential $|\Phi_{\mathrm{i}}|\to\infty$ as $r\to\infty$. Having solved equation (10), we let x_0 tend to infinity and obtain an expression for $\partial\Phi_{\mathrm{i}}/\partial x$ and $\partial\Phi_{\mathrm{i}}/\partial y$, and then obtain $\partial\Phi_{\mathrm{p}}/\partial x$ and $\partial\Phi_{\mathrm{p}}/\partial y$ by summation.

Equation (10), after transformation,

$$\frac{d^2\bar\varphi}{dy^2}+\frac{d\ln h}{dy}\frac{d\bar\varphi}{dy}-\xi^2\bar\varphi=\frac{Q}{2\sqrt{2\pi}}\left[e^{i\xi x_0}-1\right]\frac{d\ln h}{dy}\,y\,\frac{e^{-|\xi y|}}{|y|}$$

with boundary conditions $\bar\varphi\big|_{y=\pm\infty}=0$ cannot be solved by quadratures for layers acceptable in practice. In connection with this, we shall solve equation (10) by the method of successive approximations (6) and (9), each time applying the Fourier transform. As shown above, the iterative process converges for layers with $h(-\infty)/h(\infty)<e^2$ (this condition is satisfied by flow layers in radial-axial hydraulic turbines). We find φ' from the equation

$$\frac{\partial^2\varphi'}{\partial x^2}+\frac{\partial^2\varphi'}{\partial y^2}=-\frac{d\ln h}{dy}\frac{\partial\Phi_0}{\partial y},$$

applying the Fourier transform; letting x_0 tend to infinity, we obtain $\partial\Phi_{\mathrm{i}}'/\partial x$ and $\partial\Phi_{\mathrm{i}}'/\partial y$, and then $\partial\varphi_{\mathrm{p}}'/\partial x$ and $\partial\varphi_{\mathrm{p}}'/\partial y$.

In finding the solution (10), we adopted the following approximation of the layer thickness, respectively for $y \geq 0$ and $y \leq 0$:

$$\ln h^{(+)} = \ln h_{(\infty)} + \sum_{k=1}^{l_1} A_k e^{-B_k y}, \qquad \ln h^{(-)} = \ln h_{(-\infty)} + \sum_{k=1}^{l_2} C_k e^{D_k y},$$

where $B_k > 0,\ D_k > 0;\ \ln h_0^{(+)} = \ln h_0^{(-)}$ and $d \ln h^{(+)}(0)/dy = d \ln h^{(-)}(0)/dy$. We give the formula for $\partial \varphi_p' / \partial y$ for $y \geq 0$:

$$\begin{aligned}\frac{\partial \varphi_p'}{\partial y} = \frac{Q}{4t} \Bigg[\ln\left(\frac{h_{(0)}}{h}\right) & \frac{\operatorname{sh} \frac{2\pi}{t} y}{\operatorname{ch} \frac{2\pi}{t} y - \cos \frac{2\pi}{t} x} \\ & + \sum_{k=1}^{l_1} \sum_{n=1}^{m} \alpha_n A_k e^{-B_k y} \frac{\operatorname{sh} \frac{2\pi}{t}\left(y + \frac{2\beta_n}{B_k}\right)}{\operatorname{ch} \frac{2\pi}{t}\left(y + \frac{2\beta_n}{B_k}\right) - \cos \frac{2\pi}{t} x} \\ & + \sum_{k=1}^{l_2} \sum_{n=1}^{m} \alpha_n C_k \frac{\operatorname{sh} \frac{2\pi}{t}\left(y + \frac{2\beta_n}{D_k}\right)}{\operatorname{ch} \frac{2\pi}{t}\left(y + \frac{2\beta_n}{D_k}\right) - \cos \frac{2\pi}{t} x} \\ & + \frac{t}{\pi} \sum_{k=1}^{l_1} \sum_{n=1}^{m} \alpha_n A_k B_k e^{-B_k y} \ln \left\{ \frac{\operatorname{ch} \frac{2\pi}{t} y - \cos \frac{2\pi}{t} x}{\operatorname{ch} \frac{2\pi}{t}\left(y + \frac{2\beta_n}{B_k}\right) - \cos \frac{2\pi}{t} x} \right\} \Bigg],\end{aligned} \tag{11}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_m$ are determined from the expansion

$$\frac{1}{1+x} = \frac{1}{x} - \sum_{n=1}^{m} \frac{1}{x} \alpha_n e^{-\beta_n x} \qquad \text{for } x \geq 0.$$

From equation (11) it is clear that

$$\left[\frac{\partial \varphi_p'}{\partial y}\right]_{y=\infty} = \left[\frac{\partial \varphi_p'}{\partial y}\right]_{y=-\infty} = \frac{Q}{2t} \ln \left\{ \frac{h_{(0)}}{\sqrt{h_\infty h_{(-\infty)}}} \right\}.$$

The final solution for the potential of a concentrated source may be written in the form

$$\Phi_i = \Phi_{i0} + \varphi'_i + \varphi''_i + \sum_{j=2}^{\infty} \Delta\varphi''_{ij},$$

where $\Delta\varphi''_{ij}$ for $j > 2$ is determined from the equations

$$\frac{\partial^2 \Delta\varphi''_{ij}}{\partial x^2} + \frac{\partial^2 \Delta\varphi''_{ij}}{\partial y^2} = -\frac{d \ln h}{dy} \frac{\partial \varphi''_{i(j-1)}}{\partial y}.$$

V. Analogous formulas for vortices may be obtained from the equation

$$\frac{\partial^2 \psi_{\text{v}}}{\partial x^2} - \frac{d \ln h}{dy} \frac{\partial \psi_{\text{v}}}{\partial y} + \frac{\partial^2 \psi_{\text{v}}}{\partial y^2} = \Gamma \delta(x)\delta(y). \tag{12}$$

Since

$$V_{x\Gamma} = \frac{1}{h} \frac{\partial \psi_{\text{v}}}{\partial y},$$

the conditions at infinity will be

$$[hV_{x\Gamma}]_{y=\infty} = -[hV_{x\Gamma}]_{y=-\infty}.$$

VI. For layers with $h = h_1(x)h_2(y)$, the solution may be obtained as the sum of three solutions,

$$\Phi_i = \Phi_{i0} + \varphi_{ix_1} + \varphi_{iy_2},$$

where Φ_{i0} is the potential of the plane problem, and φ_{ix_1} and φ_{iy_2} are additional potentials caused by the variability of the thickness along the axes x and y. It is clear that, according to the Farrer and Keen theorem (3), the calculations given above may be applied to all layers that can be conformally transformed into the cases considered.

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Note: Figure translations are in progress. See original paper for figures.