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Abstract

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A GENERALIZED SOLUTION OF THE TRICOMI PROBLEM

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The question of the existence of a solution of the Tricomi problem for second-order mixed-type equations with lower-order terms was considered in papers (1) (for the general Lavrent' ev—Bitsadze equation) and (2). In (2) the existence of only a weak solution was proved, and its uniqueness was not proved. In the present note, by the method of finite differences, the existence of a generalized solution of the Tricomi problem for a mixed-type equation with lower-order terms is proved. A generalized solution is understood in the sense of paper (3). In (3) the uniqueness of such a solution was proved. For the first time, finite differences were used in proving the existence of solutions of boundary-value problems for a mixed-type equation in paper (4) (for the Lavrent' ev—Bitsadze equation).

Consider the equation

$$K(y)u''_{xx} + u''_{yy} + \tilde{a}(x,y)u'_x + \tilde{b}(x,y)u'_y + \tilde{c}(x,y)u = \tilde{f}(x,y), \tag{1}$$

where $K(y) = |y|^\alpha q(y)\,\mathrm{sgn}\,y,\ \alpha > 0,\ q(y) > 0,\ q(\pm 0) > 0$. By the substitution of the unknown function

$$v = u \exp\left\{\int_0^y \bar{b}(x,t)\,dt/2\right\}$$

equation (1) is reduced to the form

$$K(y)v''_{xx} + v''_{yy} + a(x,y)v'_x + c(x,y)v = f(x,y). \tag{2}$$

In what follows we shall consider only such an equation.

Let D be a domain the same as in (3). For $y < 0$ it is bounded by the characteristics Γ_0 and Γ_A of equation (2). Let, near the points $O(0,0)$, $A(x_1,0)$, the curve σ (the boundary of the domain D for $y \geq 0$) end in the arcs σ_0 and σ_A, respectively. The arc σ_0 is given by the equation $x = \lambda(y)$, $\lambda(y) \in C^{(2)}$ for $y > 0$; $\lambda(0) = \lambda'(0) = 0$. In the case $\lambda(y) \neq 0$ for $y > 0$, the inequalities

$$c_1 K' \leq \lambda' \sqrt{K} \leq c_2 K', \qquad c_1, c_2 = \mathrm{const} > 0$$

hold. The arc σ_A is situated below the straight line $y = c_3(x_1 - x)$, $c_3 = \text{const} > 0$. Denote by $B(x_B, y_B)$ the end of the curve σ_0 for $y > 0$; $B \in \sigma - \sigma_0$. We assume that in the rectangle $\Delta_B\{0 \le x \le x_B,\ 0 \le y \le y_B\}$ there are no points of the curve σ, except points of the arc σ_0.

Suppose that the coefficients of equation (2) satisfy conditions I and II of paper (^{3}) and, in addition, the following conditions.

Conditions III: 1) for any $\varepsilon > 0$, in the domain $\varepsilon < y < Y_2$, $K(y) \in C^{(1,\mu)}$, $\mu = \mu(\varepsilon)$; there exist $s_1, s_2 = \text{const} > 0$ such that

$$s_1|y|^{\alpha-1} \le K' \le s_2|y|^{\alpha-1}$$

for $y \ne 0$; 2) $a \in C^{(2)}(D^+ + D^- + \Gamma_0)$; $|y|^{2-\alpha/2}a'_y \to 0$ as $y \to -0$ in D^-; a'_x, a''_{xx} are bounded in $D^+ + D^-$, and ya'_y and $y^{2-\alpha/2}a'_y$ in D^+; 3) $c, f \in C^{(1)}(D^+ + D^-)$; c'_x, f'_x are bounded in $D^+ + D^-$, and yc'_y and yf'_y in D^+; 4) $f = 0$ on $\Gamma_0 + \sigma_0$; 5) $c + a'_x \le 0$ in $D^- + \Gamma_0$; 6) ${K'}^2 d + 16K^2 a'_x > 0$ in D^-; 7) for any sufficiently small $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ there exists $c_4 = \text{const} > 0$ such that, for $0 < y \le \varepsilon$, $0 < \varepsilon_1 \le x_1 - x \le \varepsilon_2$, $(x, y) \in D^+$, one has

$$|a| \le c_4\sqrt{K};$$

8) in the case $\lambda(y) \equiv 0$, in a neighborhood of the point O in D^+, the ratio a/x is bounded above.

Theorem 1. *Under the assumptions made, there exists a generalized solution of the Tricomi problem (^{3}) for equation (2) with boundary conditions*

$$v = 0 \text{ on } \Gamma_0 + \sigma_0, \qquad v = \varphi \text{ on } \sigma - \sigma_0,$$

where φ is a given continuous function, $\varphi(B) = 0$.

We note that the Tricomi problem for equation (2) with f and φ not necessarily equal to zero on $\Gamma_0 + \sigma_0$ reduces to the one considered, under certain assumptions on the behavior of f and φ on $\Gamma_0 + \sigma_0$.

To prove Theorem 1, introduce a mesh (see (^{3})) with sufficiently small step h, and consider difference equations of the same kind as in (^{3}) for the case $b \equiv 0$; v_h is the desired mesh function, the solution of the system of equations

$$Rv_h = f_h^* \quad \text{in } D_h, \qquad v_h = 0 \quad \text{on } \sigma_{0h} + \Gamma_{0h}, \qquad v_h = \varphi_D \quad \text{on } \sigma_h - \sigma_{0h},$$

where $f_h^* = f$ in $D_h^+ + D_h^-$, $f_h^* = 0$ on γ_h; σ_{0h} is the set of boundary nodes belonging to Δ_B.

All notation, except that introduced for the first time in this note, is as in (^{3}). Let D_0 be a domain such that $\overline{D}_0 \subset \overline{D} - \sigma + \sigma_0$; $\overline{D}_{0h}$ is the mesh domain for $\overline{D}_0$, defined analogously to $\overline{D}_h$ in (^{3}).

From estimate (12) of (^{3}) follows the uniform boundedness of v_h in $\overline{D}_h$.

The most essential part of the proof of Theorem 1 is the proof of the uniform boundedness of the first divided differences of the function v_h in $\overline{D}_{0h}$.

Using the fact that $v_{0,n} = 0$, we obtain that in $\overline{D}_{0h}$

$$v_{k,n} = \sum_{j=1}^{k+1} \beta_j^{(k,n)} v_{k+1-j,0} - \sum_{\substack{p=0,1,\dots,k \\ q=1,2,\dots,n+k-p}} l_q l_{q+1} \delta_{p,q}^{(k,n)} f_{p,q}, \tag{3}$$

where $\beta_j^{(k,n)}$, $\delta_{p,q}^{(k,n)}$ are certain numbers depending on the coefficients of the difference system of equations for $v_{k,n}$. They have the following properties:

$$\beta_j^{(k,n)} > 0, \qquad \sum_{j=1}^{k+1} \beta_j^{(k,n)} < 1, \qquad l_1' < \bar{c}\left[1 - \sum_{j=1}^{k+1} \beta_j^{(k,2)}\right],$$

$$\sum_{\substack{p=0,1,\dots,k \\ q=1,2,\dots,n+k-p}} l_q l_{q+1} \left|\delta_{p,q}^{(k,n)}\right| < y_n (3Y_1 - y_n), \tag{4}$$

where $\bar{c} = 2(Y_1 + c_*^{-1})$, and c_* is the same constant as in (11) of (^{3}).

Expressing $v_{k,2}$ by formula (3) and using (5) of (^{3}), we obtain, for $(x, 0) \in \gamma_h$,

$$\sum_{j=1}^{k+1} \beta_j^{(k)} v_h(x - jh, 0) - v_h(x, 0) + l_1' v_{hy}(x, 0) = \sum_{\substack{p=0,1,\dots,k \\ q=1,2,\dots,2+k-p}} l_q l_{q+1} \delta_{p,q}^{(k)} f_{p,q}, \tag{5}$$

where $k = x/h - 1$, $\beta_j^{(k)} = \beta_j^{(k,2)}$, $\delta_{p,q}^{(k)} = \delta_{p,q}^{(k,2)}$.

Let

$$v_{x,k,n} \equiv v_{hx}(x, -y_n), \qquad \text{where } x = kh + nh/2.$$

Putting $v_h = 0$ at the nodes of the characteristic

$$x = -h + \int_y^0 \sqrt{-K(t)}\, dt,$$

we obtain that $v_{x,-1,n} = 0$. Then the differences $v_{x,k,n}$ are expressed in terms of $v_{x,m,0}$ by means of a formula analogous to (3). Similarly to relation (5), we obtain, for $(x, 0) \in \gamma_h$, $(x + h, 0) \in \gamma_h$,

$$\sum_{j=1}^{k+1} \widetilde{\beta}_j^{(k)} v_{hx}(x-jh,0) - v_{hx}(x,0) + l_1' v_{hxy}(x,0) = \sum_{\substack{p=0,1,\dots,k\\ q=1,2,\dots,2+k-p}} l_q l_{q+1} \widetilde{\delta}_{p,q}^{(k)} \zeta_{p,q}, \quad (6)$$

where $k = x/h - 1$, $\widetilde{\beta}_j^{(k)} = \widetilde{\beta}_j^{(k,2)}$, $\widetilde{\delta}_{p,q}^{(k)} = \widetilde{\delta}_{p,q}^{(k,2)}$; $\widetilde{\beta}_j^{(k,n)}$, $\widetilde{\delta}_{p,q}^{(k,n)}$ are numbers analogous to $\beta_j^{(k,n)}$, $\delta_{p,q}^{(k,n)}$; $\xi_{p,q} = f_{x,p,q} - c_{x,p,q} v_{p+1,q} - a_{xx}(ph + \theta_1 h + qh/2, -y_q) v_{x,p,q} h/2$, $0 < \theta_1 < 1$.

For $\widetilde{\beta}_j^{(k,n)}$, $\widetilde{\delta}_{p,q}^{(k,n)}$ estimates analogous to (4) hold. We denote the left-hand side of (6) by $R_0^x v_{hx}$.

For $y > 0$ the differences v_{hx} satisfy the following equations:

$$K(v_{hx})_{xx} + (1 + a_n')(v_{hx})_{yy} + [a(x+h, y_n')(v_{hx})_x + \\ + a(v_{hx})_x]/2 + (c + a_x')v_{hx} = \zeta_h^+, \quad (7)$$

where

$$\zeta_h^+ = f_x - c_{xv} h(x+h, y_n') - a_{xx}''(x + \theta_2 h, y_n') v_{hx} h/2, \qquad 0 < \theta_2 < 1. \quad (8)$$

In (7) and (8), as also in the subsequent formulas for $y > 0$, the values of all functions for which the arguments are not indicated are taken at the point $x = kh$, $y = y_n'$; f_x, c_x, etc. are divided differences.

The uniform boundedness of the differences v_{hx}, v_{hy} at the nodes of any domain located, together with its boundary, in D^+, is proved by means of S. N. Bernstein' s method (see ([5],[6])).

By a method analogous to (7), the boundedness of the first differences on σ_{0h} for $y \geq \varepsilon > 0$ is proved. This makes it possible to prove the boundedness of v_{hx} and v_{hy} for $y \geq \varepsilon > 0$ in $\overline{D}_{0h}^{+}$ up to σ_{0h}.

Let Δ^i be the domain $\lambda(y) + ih < x < a_0$, $il_1' < y < \varepsilon$ $(a_0 < x_1,\ \varepsilon > 0,\ i = 0, 1)$; let $\overline{\Delta}_h^i$ be the mesh domain for Δ^i, defined analogously to $\overline{D}_h$ in (3); G_h^i is the set of boundary nodes for $\overline{\Delta}_h^i$. For proving the boundedness of the differences v_{hx} up to the line $y = 0$, the following is important.

Lemma 1. Suppose $|ah| < 2K$ in Δ_h^0; the function z is defined in $\overline{\Delta}_h^0$. Suppose the estimates hold: $\widetilde{\beta}_j^{(k)} > 0$, $\widetilde{\beta}_1^{(k)} + \widetilde{\beta}_2^{(k)} + \dots + \widetilde{\beta}_{k+1}^{(k)} < 1$. If

$$R_x^x z \equiv K z_{xx} + (1 + a_n') z_{yy} + [a(x+h, y_n') z_x + a z_{\overline{x}}]/2 \geq 0$$

in Δ_h^0, $R_0^x z \geq 0$ on γ_h^0 (γ_h^0 is the set consisting of the nodes with $y = 0$), then $\max z$ in $\overline{\Delta}_h^0$ is attained on $G_h^0 - \gamma_h^0$.

Applying Lemma 1 to the function

$$z_1 = (a_0 - x)^2 e^{M_1 y^2} v_{hx}^2 + M_2[v_h^2 + v_h^2(x + h, y)] + M_3 y e^y,$$

where $M_i = \text{const} > 0$ ($i = 1, 2, 3$), $x = kh$, $y = y_n'$, we prove the boundedness of v_{hx} for $0 \leq y \leq \varepsilon$, $\lambda(y) \leq x \leq x_1 - \varepsilon_1$.

Here one uses the boundedness of v_{hx} on σ_{0h} for $0 \leq y \leq \varepsilon$, proved with the help of the barrier $\omega_0 = [x - \lambda(y)](c_5 - e^{-c_6 y})$ for $y \geq 0$ and

$$\omega_0 = \left(x - \int_0^y \sqrt{-K(t)}\, dt\right)(c_5 - e^{-c_6 y})$$

for $y \leq 0$, $c_5, c_6 = \text{const} > 0$.

The boundedness of v_{hx} in $\overline{D}_{0h}^-$ follows from the fact that the differences $v_{x,k,n}$ are expressed in terms of $v_{x,m,0}$ by means of a formula analogous to (3). Moreover, in $\overline{D}_{0h}^-$ the quantities $(v_{x,k,n} - v_{x,k,n+1})/a_n$ are uniformly bounded. To prove this it is enough to apply Lemma 1 from work (3) to the functions $z^{(i)} = M_4(c_7 - e^{-y}) + (-1)^i v_{hx}$, where $i = 1, 2$; $M_4, c_7 = \text{const} > 0$.

We proceed to the estimate of the differences v_{hy} for $y \leq \varepsilon$ ($\varepsilon > 0$) in $\overline{D}_{0h}$. For $y < 0$, by $v_{y,k,n} \equiv v_{hy}(x, -y_n)$ we denote the ratio $(v_{k+1,n-2} - v_{k,n})/(l_{n-1} + l_n)$, and by $v_{l,k,n}$ the ratio $(v_{k,n-1} - v_{k,n})/l_n$ (the "oblique" difference). Using the fact that v_h satisfies system (3) from work (3) and that $v_{l,0,m} = 0$, we obtain:

$$v_{l,k,n} = \sum_{i=0}^{k-1} \left[\frac{a_{n+i} - \alpha_{k-i,n+i}}{1 - a_{n+i}} \, \frac{h}{l_{n+i}} \, v_{x,k-i-1,n+i} \right.$$
$$\left. - \frac{l_{n+i} + l_{n+i+1}}{2} (c_{k-i,n+i} v_{k-i,n+i} - f_{k-i,n+i}) \right]. \qquad (9)$$

From (9), in view of the boundedness of v_h and v_{hx}, there follows the boundedness of the "oblique" differences. The boundedness of the differences v_{hy} in $\overline{D}_{0h}$ follows from the fact that they are expressed in terms of v_{hx} and $v_{l,k,n}$.

In D_{0h}^+, for $0 < y \leqslant \varepsilon$, we first prove the boundedness of $\sqrt{y}\, v_{\bar{h}y}$. In doing so we use the fact that

$$R_1 z_2 \equiv K z_{2xx} + (1 + a_n') z_{2yy} + a(z_{2x} + z_{2\bar{x}})/2 \geqslant 0$$

in Δ_h^1, for

$$z_2 = (a_0 - x)^2 y_n v_{hy}^2 + v_{hx}^2 + (y_n')^2 v_{hx}^2(x, y_{n-1}) + M_5[v_h^2 + v_h^2(x-h, y_n') + v_h^2(x+h, y_n')] + M_6[1 - (y_n')^r],$$

$$r = \min(\alpha, 1/24), \qquad M_5, M_6 = \mathrm{const} > 0,$$

and, consequently, $\max z_2$ in $\overline{\Delta}_h^1$ is attained on the boundary G_h^1. Similarly we prove the boundedness of v_{hy}, taking

$$z_3 = (a_0 - x)^2 v_{hy}^2 + M_7(1 - \sqrt{y_n'})$$

and using the fact that in Δ_h^0

$$R_h'' z_3 \equiv R_1 z_3 - \nu_h z_{3y} \geqslant 0,$$

where

$$\nu_h = (a_{n-1}' - a_n')/l_n' + (1 + a_{n-1}) K_{\bar{y}}/K(y_{n-1}).$$

Thus, in $\overline{D}_{0h}$ the first differences of v_h are uniformly bounded.

Next, from the values of v_h we define bilinear functions v_h' and piecewise-constant ones (similarly to (8), Chap. I, § 6). On the basis of Arzelà' s theorem one can select a subsequence v_{h_m}' uniformly convergent in D_0 to a certain continuous function v. This function satisfies requirements 1)–5) of the generalized solution of the Tricomi problem (^{3}). In proving this, the boundedness of the first differences of v_h and the boundedness of the quantities

$$(v_{x,k,n} - v_{x,k,n+1})/a_n$$

are essentially used. The fact that v satisfies the integral identity (14) of (^{3}) ($b = 0$) is proved similarly to (8), Chap. III. The continuity of v at points of the curve $\sigma - \sigma_0$ and the satisfaction of the condition $v = \varphi$ on $\sigma - \sigma_0$ are ensured by the regularity of the points of the curve $\sigma - \sigma_0$ (see (^{9}), p. 106; (^{10}), p. 254). For the point A, the function

$$\omega_A = c_8(x_1 - x) - y - c_9 y^2,$$

where $c_8, c_9 = \mathrm{const} > 0$, may be used as a barrier. Thus, Theorem 1 is proved.

From the uniqueness of the generalized solution of the Dirichlet problem (understood in a sense analogous to (^{3})) in the domain situated in the half-plane

$y > 0$, and the existence of a smooth solution of the Dirichlet problem (see ([9])), it follows that v in D^+ has continuous derivatives of the first and second order which enter equation (2), and satisfies (2) in the usual sense.

From Theorem 1 and the results of ([3]) it follows that, under the fulfillment of conditions I, II (see ([3])) and III, there exists and is unique a generalized solution of the Tricomi problem for equation (2).

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Note: Figure translations are in progress. See original paper for figures.