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1965

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Abstract

Full Text

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ON THE APPROXIMATION OF FUNCTIONS BY POLYNOMIALS WITH CONSTRAINTS

(Presented by Academician S. N. Bernstein on 2 XII 1964)

The problem of approximating functions by algebraic polynomials and trigonometric polynomials whose coefficients are connected by a linear relation was first studied by V. A. Markov ([5]). The paper ([5]) initiated numerous investigations in this direction (see ([3,4])).

The present paper is devoted to Jackson-type estimates for best approximations of a continuous function in the case of homogeneous linear constraints of a certain special form. For the same constraints, the paper indicates the exact order of the upper bound of best approximations over a class of functions analytic and bounded in a prescribed domain. For approximations without constraints, the latter problem was solved by N. A. Akhiezer (([1]), pp. 230–235).

§ 1. Let X be a linear normed space and $\{x_k\}_0^\infty \subset X$. By $X_n = \{p_n\}$ $(n = 0, 1, ...)$ we denote the subspace of X spanned by the system $\{x_k\}_0^n$. Let Ψ be a homogeneous additive functional defined on the union of all X_n, and let its norm on X_n be equal to $\|\Psi\|_n$. For any element $f \in X$ put

$$E_n(f) = \inf_{p_n \in X_n} \|f - p_n\| = \|f - p_n(f)\|, \qquad E_n(f; \Psi) = \inf_{\Psi(p_n)=0} \|f - p_n\|,$$

where the norm is the same as in the space X. In this section we agree to regard $0/0 = 0$. In these notations the following inequalities are valid*:

$$\begin{aligned} {}^1/_3\{E_n(f) + |\Psi(p_n(f))|(\|\Psi\|_n)^{-1}\} \le E_n(f; \Psi) \le \\ \le E_n(f) + |\Psi(p_n(f))|(\|\Psi\|_n)^{-1} \qquad (n = 0, 1, ...); \end{aligned} \tag{1.1}$$

$$\begin{aligned} E_n(f; \Psi) \le |\Psi(p_0(f))|(\|\Psi\|_n)^{-1} + \\ +2\left\{E_n(f) + (\|\Psi\|_n)^{-1} \sum_{k=0}^{n-1} E_k(f)\|\Psi\|_{k+1}\right\} \qquad (n = 0, 1, ...). \end{aligned}$$

§ 2. Let $X = C_{[-1,1]}$; $X_n \subset X$ is the subspace of algebraic polynomials $p_n(x)$ of degree n. We shall denote the functional Ψ by $\Psi_{\alpha,a}$ $(\alpha > -1,\ a > 1)$, if it can be represented in the form

$$\Psi(p_n) = \int_{-a}^{a} p_n(x)(a^2 - x^2)^\alpha \varphi(x)\, dx = \Psi_{\alpha,a}(p_n), \tag{2.1}$$

where $\varphi(x)$ is a function summable on the interval $[-a, a]$, continuous at the points $x = \pm a$, and $\varphi^2(a)+\varphi^2(-a) \neq 0$. For $a > 1$ put $g(a) = a+(a^2-1)^{1/2} > 1$.

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* A special case of inequality (1.1) was indicated earlier ([6]).

Let $\omega(\delta; f)$ be the modulus of continuity of the function $f(x) \in C_{[-1,1]}$. From the estimate (1,2), inequality (3) of the work [7], and D. Jackson' s inequality

$$E_n(f) \leqslant \frac{c_r}{n^r}\,\omega\left(\frac{1}{n};\, f^{(r)}\right) \qquad (n > r) \tag{2,2}$$

there follows the following

Theorem 1. Let $r \geqslant 0$ be an integer, $\alpha > -1$, $a > 1$. If $f^{(r)}(x) \in C_{[-1,1]}$, then for $n > r$

$$E_n(f; \Psi_{\alpha,a}) \leqslant \frac{K_1}{n^r}\,\omega\left(\frac{1}{n};\, f^{(r)}\right) + K_2\|f\|_{C[-1,1]}\,g^{-n/2}(a), \tag{2,3}$$

where the constants K_1 and K_2 depend only on r and $\Psi_{\alpha,a}$.

Let Γ_b be the ellipse with foci at the points $x = \pm 1$, passing through the point $b > 1$; let $G(b)$ be the finite domain in the z-plane with boundary Γ_b. We shall say that a function $f(z)$ analytic in the domain $G(b)$ belongs to $MB(b)$ if it is real on the segment $[-1, 1]$ and $|f(z)| \leqslant M$ for $z \in G(b)$.

Theorem 2. Let $\alpha > -1$, $a > 1$, $b > 1$. For every functional $\Psi_{\alpha,a}$ of the form (2,1) there exists a constant $K_3 = K_3(b, \Psi_{\alpha,a})$ such that, if the function $f(z) \in MB(b)$, then

$$E_{n-1}(f; \Psi_{\alpha,a}) \leqslant \begin{cases} K_3 M g^{-n}(b), & \text{if } b < a, \\ K_3 M n^{\alpha+1} g^{-n}(a), & \text{if } b \geqslant a, \end{cases} \tag{2,4}$$

$$(n = 1, 2, \ldots).$$

For every number $b > 1$ in the class $MB(b)$, inequality (2,4) is exact in order as $n \to \infty$.

Let $h(x)$ $(-a \leqslant x \leqslant a)$ be a piecewise-constant function of bounded variation such that the endpoints of the segment $[-a, a]$ are not limit points for the discontinuities of $h(x)$, and

$$[h(a) - h(a-0)]^2 + [h(-a+0) - h(-a)]^2 \neq 0.$$

Put

$$D_a(p_n) = \int_{-a}^{a} p_n(x)\, dh(x) \qquad (n = 0, 1, ...). \tag{2,5}$$

For $a > 1$, inequality (2,3) is preserved if in it $\Psi_{\alpha,a}$ is replaced by D_a. Therefore, for the best approximations $E_n(f; D_a)$ we shall formulate only an analogue of Theorem 2.

Theorem 3. Let $a > 1$, $b > 1$, and let D_a be a functional of the form (2,5). Then there exists a constant $K_4 = K_4(b, D_a)$ such that, if $f(z) \in MB(b)$, then

$$E_n(f; D_a) \leqslant \begin{cases} K_4 M g^{-n}(b), & \text{if } b < a, \\ K_4 M \ln(n+2)\, g^{-n}(a), & \text{if } b = a, \\ K_4 M g^{-n}(a), & \text{if } b > a, \end{cases} \tag{2,6}$$

$$(n = 0, 1, ...).$$

For every number $b > 1$ in the class $MB(b)$, inequality (2,6) is exact in order as $n \to \infty$.

Let us note that Theorem 3 contains an important special case, when

$$D_a(p_n) \equiv p_n(a), \qquad a > 1.$$

§ 3. Let $0 < a < b \leqslant \pi$; $X = C_{[-a,a]}$; $X_{2n} \subset X$ be the subspace of trigonometric polynomials $t_n(u)$ of order n. The best approximations $E'_{2n}(f)$ and $E_{2n}(f; \Psi)$ (see § 1) in this case will be denoted by $E_n(f, a)$ and $E_n(f, a, \Psi)$, respectively. Put

$$\Psi(t_n) = \Psi_{a,b}(t_n) = \int_{-b}^{b} t_n(u)(b^2 - u^2)^\alpha \varphi(u)\, du, \tag{3,1}$$

where $\varphi(u)$ is a function summable on the segment $[-b, b]$, continuous at the points $u = \pm b$, and

$$\varphi^2(b) + \varphi^2(-b) \neq 0.$$

In estimating the quantity $E_n(f, a, \Psi_{\alpha,\pi})$ for analytic functions, it is useful to distinguish two cases according as $\varphi(u)$ does or does not satisfy the equality

$$\varphi(\pi) + \varphi(-\pi) = 0. \tag{3,2}$$

If this equality holds, we shall also require that in neighborhoods of the points $u = \pm\pi$ the inequalities

$$|\varphi(u) - \varphi(\pm\pi)| \leq K(\pi \mp u) \tag{3,3}$$

hold, respectively.

Let $q_1(w) = (1 - \cos a)^{-1}(2\cos w - 1 - \cos a)$ and $g_1(w) = g(q_1(w))$. For any fixed number $\lambda > 0$, denote by $D(a, \lambda)$ the finite domain lying in the strip $-\pi \leq \operatorname{Re} w \leq \pi$ and bounded by the level line $|g_1(w)| = g_1(i\lambda) > 1$.

Let the number λ_b be the root of the equation $g_1(i\lambda_b) = |g_1(b)|$ $(a < b \leq \pi)$. For $\lambda > \lambda_\pi$, the boundary of the domain $D(a, \lambda)$ will include segments of the straight lines $w = \pm\pi + iv$. We shall say that a function $f(w)$, analytic in the domain $D(a, \lambda)$, belongs to $MB(a, \lambda)$ if it is real on the interval $[-a, a]$ and $|f(w)| \leq M$ for $w \in D(a, \lambda)$. In this case, when $\lambda > \lambda_\pi$, we shall assume that the function $f(w)$, extended with period 2π, remains analytic in the corresponding domain.

Using a result of N. I. Akhiezer ([1], p. 235), one can prove the following assertion:

Lemma. If $f(w) \in MB(a, \lambda)$, then

$$E_n(f, a) \leq HMg_1^{-n}(i\lambda) \quad (n = 0, 1, ...), \tag{3,4}$$

where the constant H does not depend on $f(w)$. In the class $MB(a, \lambda)$, the estimate (3,4) is sharp in order as $n \to \infty$.

Hence, with the aid of inequality (1,1) and inequality (1)–(2) of paper [7], the following theorem follows:

Theorem 4. Let $0 < a < b \leq \pi$, $\alpha > -1$, $\lambda > 0$, and let $\Psi_{\alpha,b}$ be a functional of the form (3,1). Then there exists a constant $H_1 = H_1(a, \lambda, \Psi_{\alpha,b})$ such that, for any function $f(w) \in MB(a, \lambda)$ and any $n = 1, 2, ...$, we have:

1) if $a < b \leq \pi$ and $\lambda < \lambda_b$, then

$$E_{n-1}(f, a, \Psi_{\alpha,b}) \leq H_1 M g_1^{-n}(i\lambda);$$

2) if $a < b < \pi$ and $\lambda \geq \lambda_b$, then

$$E_{n-1}(f, a, \Psi_{\alpha,b}) \leq H_1 M n^{\alpha+1} g_1^{-n}(i\lambda_d);$$

3) if $b = \pi$, $\lambda \geq \lambda_\pi$, and the functional $\Psi_{\alpha,\pi}$ does not satisfy condition (3,2), then

$$E_{n-1}(f, a, \Psi_{\alpha,\pi}) \leq H_1 M n^{(\alpha+1)/2} (\operatorname{ctg} a/4)^{-2n};$$

4) if $b = \pi$, $\lambda \geq \lambda_\pi$, and the functional $\Psi_{\alpha,\pi}$ satisfies conditions (3,2) and (3,3), then

$$E_{n-1}(f, a, \Psi_{\alpha,\pi}) \leq H_1 M n^{(\alpha+2)/2} (\operatorname{ctg} a/4)^{-2n}.$$

For any number $\lambda > 0$, all these inequalities in the class $MB(a, \lambda)$ are sharp in order as $n \to \infty$.

For continuous functions, with the aid of a theorem of S. B. Stechkin (see, for example, [2], p. 113), one can prove the following assertion:

Theorem 5. Let $0 < a < d \leq \pi$ and $f(x) \in C[-d, d]$. For every functional $\Psi_{\alpha,b}$ $(\alpha > -1,\ a < b \leq \pi)$ of the form (3,1), there exists a constant $C_k = C_k(a, \Psi_{\alpha,b}, \|f\|_{C[-d,d]})$ and a number $N_k(a)$ such that for $n > N_k(a)$ we have:

$$E_n(f, a, \Psi_{\alpha,b}) \leq C_k \omega_k(1/n, f, a),$$

where $\omega_k(\delta, f, a)$ is the modulus of continuity of the function $f(x)$ of order $k = 1, 2, \ldots$ on the interval $[-a, a]$.

Denote by $E_n^{(0)}(f, a, m)$ and $E_n^{(1)}(f, a, m)$ the best approximations of the function $f(x)$ by trigonometric polynomials $t_n(x)$ that do not contain $\cos mx$ or $\sin mx$, respectively.

Corollary 1. Let $0 < a < d \leqslant \pi$, $l = 0, 1$, $f(x) \in C[-d, d]$.
For any integer $m \geqslant 0$ there exist a constant $C_k^{(l)} = C_k^{(l)}(a, m, \|f\|_{C[-d,d]})$ and an $N_k(a)$ such that, for $n > N_k(a)$, we have

$$E_n^{(l)}(f, a, m) \leqslant C_k^{(l)} \omega_k(1/n, f, a).$$

Corollary 2. Let $0 < a < \pi$, $\lambda > 0$, $l = 0, 1$, $f(w) \in MB(a, \lambda)$.
For every integer $m \geqslant l$ there exist constants $H_m^{(l)} = H_m^{(l)}(a, \lambda)$ such that, for $n > m$, we have

$$E_n^{(l)}(f, a, m) \leqslant \begin{cases} H_m^{(l)} M g_1^{-n}(i\lambda), & \text{if } \lambda < \lambda_\pi, \\ H_m^{(l)} M n l^{1/2} (\operatorname{ctg} a/4)^{-2n}, & \text{if } \lambda \geqslant \lambda_\pi \end{cases} \qquad (l = 0, 1).$$

For any $\lambda > 0$ these inequalities in the class $MB(a, \lambda)$ are sharp in order as $n \to \infty$.

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Received
23 XI 1964

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.