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Abstract

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MATHEMATICS

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CONVERGENCE OF FORMULAS OF APPROXIMATE INTEGRATION ON FUNCTIONS FROM $L_2^{(m)}$

In the author' s papers (^{1,2}) it was established that the extremal function $u(x)$, i.e., the function on which the error functional attains its greatest value on the sphere of unit radius in $L_2^{(m)}$, is a solution of the polyharmonic equation with right-hand side

$$\Delta^m u = (-1)^{m+1} l(x). \tag{1}$$

We shall consider periodic functions of the variable, defined on a certain torus Ω, and a system of nodes of a cubature formula of the form

$$x^{(\gamma)} = hH\gamma, \tag{2}$$

where $x^{(\gamma)}$ is a column vector of the coordinates of a point; γ is an integer column vector; H is a matrix with determinant one; h is a small parameter. Suppose, moreover, that the periods of the torus Ω, which we do not consider, are multiples of the columns of the matrix hH, i.e., of the periods of the lattice.

Theorem 1. *Among all coefficients C_γ entering into the expression for the error functional*

$$l(x) = 1 - \sum_\gamma h^n \delta(x - x^{(\gamma)}), \tag{3}$$

the smallest value of $l(x)$ in the norm of $L_2^{(m)}$ is given by the constants

$$C_\gamma = 1. \tag{4}$$

The proof of this theorem, like its formulation itself, though for other spaces, is known. As is known, the norm $l(x)$ is a strictly convex function of C_γ, i.e.,

$$\left\| \frac{l_1(x) + l_2(x)}{2} \right\| < C, \tag{5}$$

if

$$\|l_1(x)\| = \|l_2(x)\| = C.$$

If the coefficients of some functional of the form (3) are not all equal to one another, then the functionals $l(x)$ and $l(x - hH\gamma)$, with the same nodes, do not coincide; moreover, their half-sum will be a functional with the same nodes, but smaller in norm. Consequently, $l(x)$ cannot be minimal. The theorem is proved.

The Fourier method makes it possible to give an explicit expression for the extremal function of the extremal functional

$$l_0(x) = 1 - \sum_{\gamma} h^n \delta(x - hH\gamma) \tag{6}$$

in the form

$$u(x) = (2\pi)^{-2m} h^{2m} \sum_{\beta \neq 0} \frac{e^{-2\pi i \beta h^{-1} H^{-1} x}}{[A(\beta)]^m}, \tag{7}$$

where $A(\beta)$ is the quadratic form with matrix

$$A = H^{-1} H^{-1*}. \tag{8}$$

This expression already makes it possible, in an elementary way, to obtain an explicit expression for the norm of the extremal functional

$$\|l_0(x)\| = (2\pi)^{-m} h^m \sqrt{\Omega} \sqrt{\xi\left(H^{-1}, 2m\right)}, \tag{9}$$

where

$$\xi\left(H^{-1}, 2m\right) = \sum_{\gamma \neq 0} \frac{1}{[A(\gamma)]^m}. \tag{10}$$

From (9) there follows, valid for all $L_2^{(m)}$, the estimate of the magnitude of the error of the cubature formula

$$|(l,\varphi)| \le (h/2\pi)^m \sqrt{\Omega}\sqrt{\xi\left(H^{-1},2m\right)}\|\varphi\|_{L_2^{(m)}}. \tag{11}$$

The aim of the present note is the following theorem:

Theorem 2. For each individual function $\varphi(x) \in L_2^{(m)}$, as $h \to 0$, the sharper estimate holds

$$|(l,\varphi)| \le (h/2\pi)^m \sqrt{\xi\left(H^{-1},2m\right)}\|\varphi\|_{L_1^{(m)}} + o(h^m), \tag{12}$$

where $o(h^{(m)})$ depends on the choice of the function $\varphi(x)$ and

$$\|\varphi\|_{L_2^{(m)}} = \int \left[\sum_{|\alpha|=m} (D^\alpha\varphi)^2\right]^{1/2} dx. \tag{13}$$

As follows from the theorem, estimate (11), while completely sharp for the entire class of functions in $L_2^{(m)}$ periodic on the torus Ω, is not sharp for any single concrete function. Although equality in (11) is attained for any prescribed h and corresponding $\varphi(x)$, since for all functions except solutions of the equation

$$\sum_{|\alpha|=m} (D^\alpha\varphi)^2 = \text{const}, \tag{14}$$

the strict inequality holds

$$\|\varphi\|_{L_m^{(1)}} < \sqrt{\Omega}\|\varphi\|_{L_2^{(m)}}, \tag{15}$$

and the extremal function (7) does not satisfy this equation, it follows that, for sufficiently small h, estimate (12) is stronger for any concrete function than estimate (11).

Let us give the idea of the proof of the main theorem. The magnitude of the error $(l(x),\varphi(x))$ for any function $\varphi(x) \in L_2^{(m)}$ is expressed by the formula

$$(l(x),\varphi(x)) = (\Delta^m u,\varphi) = \sum_{|\alpha|=m} (D^\alpha u, D^\alpha \varphi) = \int_\Omega \sum_{|\alpha|=m} D^\alpha u \, D^\alpha \varphi \, dx. \tag{16}$$

Divide the whole volume Ω into elementary parallelepipeds Ω_γ with sides expressed by the columns of the matrix hH, and with origin at $hH\gamma$. Applying Bunyakovsky’ s inequality, we shall have, after carrying out the corresponding calculations,

$$(l(x), \varphi(x)) = \sum_{\gamma} \int_{\Omega_\gamma} \sum_{|\alpha|=m} D^\alpha \varphi \, D^\alpha u \, dx \leq$$

$$\leq \sum_{\gamma} \left[\int_{\Omega_\gamma} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 dx \right]^{1/2} \left[\int_{\Omega_\gamma} \sum_{|\alpha|=m} (D^\alpha u)^2 dx \right]^{1/2} =$$

$$= \left(\frac{h}{2\pi} \right)^m \sqrt{\xi\left(H^{-1}, 2m\right)} \sum_{\gamma} h^{n/2} \left\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 dx \right\}^{1/2} . \tag{17}$$

To prove estimate (12), it remains to show that

$$\sum_{\gamma} h^{n/2} \left\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 \, dx \right\}^{1/2} = \int_{\Omega} \left\{ \sum_{|\alpha|=m} (D^\alpha \varphi)^2 \right\}^{1/2} dx + o(1). \tag{18}$$

Consider the function

$$f_\gamma(\lambda) = \int_{\Omega_\gamma} \left(\left[\sum_{|\alpha|=m} (D^\alpha \varphi)^2 \right]^{1/2} - \lambda \right)^2 dx. \tag{19}$$

The expression $f_\gamma(\lambda)$ is a positive quadratic trinomial with respect to λ

$$f_\gamma(\lambda_\gamma) = h^n \lambda^2 - 2\lambda \int_{\Omega_\gamma} \left[\sum_{|\alpha|=m} (D^\alpha \varphi)^2 \right]^{1/2} dx + \int_{\Omega_\gamma} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 \, dx. \tag{20}$$

For all positive trinomials of the form $f(\lambda) = a\lambda^2 - 2b\lambda + c$, the equality

$$a \min f(\lambda) = ac - b^2 \tag{21}$$

holds. Let

$$\min f_\gamma(\lambda) = f_\gamma(\lambda_\gamma) = \varepsilon_\gamma. \tag{22}$$

Since the function $\left[\sum_{|\alpha|=m} (D^\alpha \varphi)^2\right]^{1/2}$ will obviously be summable with square, it is possible to approximate it in norm by means of the step function

$$\psi(x) = \lambda_\gamma \quad (x \in \Omega_\gamma). \tag{23}$$

It follows that the expression

$$\sum_\gamma \varepsilon_\gamma = \varepsilon = \tau_\varphi^{(m)}(h) \tag{24}$$

will tend to zero as $h \to 0$. From (21) and (22), however, it follows that

$$\varepsilon_\gamma = \int_{\Omega_\gamma} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 \, dx - \frac{1}{h^n} \left(\int_{\Omega_\gamma} \left[\sum_{|\alpha|=m} (D^\alpha \varphi)^2 \right]^{1/2} dx \right)^2 , \tag{25}$$

and, consequently,

$$\sum_\gamma h^{n/2} \left\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 \, dx \right\}^{1/2} = \sum_\gamma h^{n/2} \left\{ \frac{1}{h^n} \left(\int_{\Omega_\gamma} \left[\sum_{|\alpha|=m} (D^\alpha \varphi)^2 \right]^{1/2} dx \right)^2 + \varepsilon_\gamma \right\}^{1/2} =$$

$$= \sum_\gamma \left\{ \left(\int_{\Omega_\gamma} \left[\sum_{|\alpha|=m} (D^\alpha \varphi)^2 \right]^{1/2} dx \right)^2 + h^n \varepsilon_\gamma \right\}^{1/2} . \tag{26}$$

Finally, from the triangle inequality it follows that

$$\left\{ \left(\int_{\Omega_\gamma} \left[\sum_{|\alpha|=m} (D^\alpha \varphi)^2 \right]^{1/2} dx \right)^2 + h^n \varepsilon_\gamma \right\}^{1/2} - \int_{\Omega_\gamma} \left[\sum_{|\alpha|=m} (D^\alpha \varphi)^2 \right]^{1/2} dx \le h^{n/2} \sqrt{\varepsilon_\gamma} \tag{27}$$

and, further,

$$\sum_\gamma h^{n/2} \sqrt{\varepsilon_\gamma} \le \left(\sum_\gamma h^n \right)^{1/2} \left(\sum_\gamma \varepsilon_\gamma \right)^{1/2} = \sqrt{\Omega} \sqrt{\tau_\varphi^{(m)}(h)}. \tag{28}$$

Summing (27) and using (26) and (28), we obtain (18). The theorem is proved.

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CITED LITERATURE

1. S. L. Sobolev, DAN, **137**, No. 3, 527 (1961).

2. S. L. Sobolev, *Lectures on the Theory of Cubature Formulas.* Novosibirsk, 1964.

Note: Figure translations are in progress. See original paper for figures.