D. L. BERMAN

MATHEMATICS

1965

SovietRxiv

Abstract

Full Text

D. L. BERMAN

ON THE THEORY OF INTERPOLATION

(Presented by Academician S. N. Bernstein on 4 XII 1964)

MATHEMATICS

1°. We shall say that a system of points

$$-1 \le x_n^{(n)} < x_{n-1}^{(n)} < \cdots < x_1^{(n)} \le 1, \qquad n = 1, 2, \dots, \tag{1}$$

is quasi-uniformly distributed in the segment $[-1, 1]$ if there exist positive constants C_1 and C_2 such that the inequalities

$$C_1/n \le \theta_{k+1}^{(n)} - \theta_k^{(n)} \le C_2/n, \qquad k = 0, 1, 2, \dots, n; \quad n = 1, 2, \dots,$$

hold, where $x_k^{(n)} = \cos\theta_k^{(n)},\ \theta_0 = 0,\ \theta_{n+1} = \pi$.

The following general result is known (^{1}). If the n-th row of matrix (1) consists of the zeros of an orthogonal polynomial of degree n, whose weight satisfies the inequalities $0 < m \le p(x)\sqrt{1-x^2} \le M,\ -1 \le x \le 1$, then the points of such a matrix are quasi-uniformly distributed in $[-1, 1]$. The results of the works (^{1}-^{3}) show that quasi-uniformly distributed nodes are of considerable interest for the theory of interpolation. Among them a special role is played by the P. L. Chebyshev nodes

$$x_k^{(n)} = \cos\frac{2k-1}{2n}\pi, \qquad k = 1, 2, \dots, n; \quad n = 1, 2, \dots,$$

for which the overwhelming majority of results in interpolation theory have been proved. It should apparently be regarded as natural to attempt to extend the results established for Chebyshev nodes to the general case of quasi-uniformly distributed nodes. The results of this note, in particular, show that such an extension is not always possible.

2°. As the system of nodes (1) we take the points

$$x_0^{(n+2)} = 1, \qquad x_{n+1}^{(n+2)} = -1, \qquad x_k^{(n+2)} = \cos\frac{2k-1}{2n}\pi, \qquad 1 \le k \le n; \quad n = 1, 2, \dots. \tag{2}$$

Obviously, these points are quasi-uniformly distributed in the segment $[-1,1]$. At these nodes we construct the Lagrange and Hermite-Fejér interpolation processes for the function $f(x)=|x|$. In this connection let us recall the unexpected result of S. N. Bernstein (^{5}), according to which the Lagrange interpolation process constructed for the same function at equidistant nodes of the interval $[-1,1]$ diverges at every point $x \neq 0$ of $(-1,1)$.

Theorem 1. *The Lagrange interpolation process constructed for the function $f(x)=|x|$ at the nodes (2) diverges at the point $x=0$.*

Let us outline the proof. After simple calculations we obtain that, for an even function at the nodes (2), the Lagrange interpolation polynomial has the form

$$L_n(f,x)=f(1)T_n(x)+\sum_{j=1}^{2p} f(x_j)\,\frac{2x_j(x^2-1)T_n(x)}{(x_j^2-1)(x^2-x_j^2)T_n'(x_j)},$$

where $T_n(x)=\cos n\arccos x$ and $n=4p$. Hence it follows that, for $f(x)=|x|$,

$$L_n(|x|,0)=2\sum_{j=1}^{2p} l_j(-1), \tag{3}$$

where $\{l_j(x)\}$ are the fundamental Lagrange polynomials constructed at the Chebyshev nodes. One can prove that

$$\sum_{j=1}^{2p} l_j(-1) > \frac{1}{\pi}-\frac{1}{n}. \tag{4}$$

From (3) and (4) it follows that

$$L_n(|x|,0) > \frac{2}{\pi}-\frac{2}{n}.$$

Thus, the process diverges at $x=0$. It is curious that this same process, constructed at the Chebyshev nodes, converges at every point of the interval $[-1,1]$, and the convergence is even uniform.

Theorem 2. Let $f(x)=|x|\sqrt{1-x^2}\varphi(|x|)$, where $\varphi(x)$ is an arbitrary function continuous on the segment $[-1,1]$. Then the process $\{L_n(f,x)\}_{n=1}^{\infty}$, constructed at the nodes (2), converges at the point $x=0$.

Denote by $H_n(f,x)$ the Hermite-Fejér interpolation polynomial of degree $(2n-1)$, which is uniquely determined by the conditions

$$H_n(f,x_k)=f(x_k),\qquad H_n'(f,x_k)=0,\qquad k=1,2,\dots,n.$$

Theorem 3. The Hermite-Fejér interpolation process $\{H_n(f)\}_{n=1}^{\infty}$, constructed for the function $f(x) = |x|$ at the nodes (2), satisfies the equality

$$\lim_{n\to\infty} H_n(f, 0) = \infty.$$

We outline the proof. For the nodes (2),

$$\begin{aligned} H_{n+2}(f, x) = {} & \frac{f(1)}{2}\,[1 - (2n^2 + 1)(x - 1)](x + 1)T_n(x) \\ & - \frac{f(-1)}{2}\,[1 + (2n^2 + 1)(x + 1)](x - 1)T_n(x) \\ & + \sum_{j=1}^{n} f(x_j)\left(1 + \frac{3x_j(x - x_j)}{1 - x_j^2}\right)\frac{4x_j^2(x^2 - 1)^2 T_n^2(x)}{(x^2 - x_j^2)^2(x_j^2 - 1)[T_n'(x_j)]^2}. \end{aligned}$$

Therefore, for even f,

$$H_{n+2}(f, 0) = f(1)(n^2 + 1) + \frac{2}{n^2}\sum_{j=1}^{2p}\frac{1 - 4x_j^2}{x_j^2(1 - x_j^2)^2}, \qquad n = 4p.$$

In particular, for $f(x) = |x|$ we obtain that

$$\begin{aligned} H_{n+2}(|x|, 0) = (n^2 + 1) + \frac{2}{n^2}\sum_{j=1}^{2p}\frac{1}{\cos\varphi_j} + \frac{2}{n^2}\sum_{j=1}^{2p}\frac{\cos\varphi_j}{\sin^2\varphi_j} \\ + \frac{3}{8n^2}\sum_{j=1}^{2p}\frac{1}{\cos^4(\varphi_j/2)} - \frac{3}{8n^2}\sum_{j=1}^{2p}\frac{1}{\sin^4(\varphi_j/2)}, \end{aligned}$$

where $x_j = \cos\varphi_j$ and $\varphi_j = (2j - 1)\pi/2n$. Therefore

$$H_{n+2}(|x|, 0) > n^2 + 1 - \frac{3}{8n^2}\sum_{j=1}^{2p}\frac{1}{\sin^4(\varphi_j/2)}. \tag{5}$$

To complete the proof a lemma is needed.

Lemma. The identity holds

$$\sum_{j=1}^{2n}\frac{1}{\sin^4\varphi_j/2} = 4n^4 - \frac{2}{3}n^3 - \frac{8}{3}n^2. \tag{6}$$

In proving the lemma we used the well-known identity of M. Riesz (4). It is clear that from (5) and (6) there follows the inequality

$$H_{n+2}(|x|, 0) > n^2/4 + n/8 + 1/2.$$

Thus, Theorem 3 is proved.

It is known that the same process, constructed at the Chebyshev nodes, converges uniformly on $[-1, 1]$.

3°. We have already mentioned that in the proof of Theorem 3 an identity of M. Riesz was used, which plays a very important role in the constructive theory of functions. This identity admits the following generalization.

Denote by Π_n the set of all trigonometric polynomials of order $\leq n$. Let

$$\Phi_n(t) = \sum_{k=0}^{n} r_k \sin(kt + \alpha_k).$$

Put

$$\widetilde{\Phi}_n(t) = r_n + 2\sum_{k=0}^{n-1} r_k \cos[(n-k)t + \alpha_n - \alpha_k].$$

Theorem 4. If $t_n \in \Pi_n$, then the identity holds

$$\int_0^{2\pi} t_n(x+\theta)\Phi_n(\theta)\, d\theta = \frac{\pi}{2n}\sum_{r=1}^{2n} t_n\left(x + \varphi_r - \frac{\alpha_n}{n}\right)(-1)^{r-1}\widetilde{\Phi}_n\left(\varphi_r - \frac{\alpha_n}{n}\right), \quad (7)$$

where

$$\varphi_r = \frac{2r-1}{2n}\pi.$$

Regarding the proof of this identity, let us note that it suffices to verify it only for each of the functions $\{\cos kx\}_{k=0}^{n}$, $\{\sin kx\}_{k=1}^{n}$. The case is especially interesting when

$$\Phi_n(t) = \frac{1}{\pi}\left(\frac{\sin(n+1/2)t}{2\sin t/2}\right), \quad (8)$$

then

$$\widetilde{\Phi}_n(t) = \frac{\sin^2 nt/2}{\pi \sin^2 t/2}.$$

Therefore equality (7) passes into the identity of M. Riesz (4)

$$t_n'(x) = \frac{1}{2n} \sum_{r=1}^{2n} t_n(x+\varphi_r) \frac{(-1)^{r-1}}{2 \sin^2 \varphi_r/2}.$$

With the aid of this identity M. Riesz obtained a very elegant proof of the classical theorem of S. N. Bernstein, according to which

$$\|t_n'\| \leq n\|t_n\|.$$

We shall suppose that in Π_n a norm has been introduced possessing the usual properties of a norm, and also the additional property that

$$\|t_\alpha\| \leq \|t\|, \qquad -\infty < \alpha < \infty, \qquad t \in \Pi_n,$$

where $t_\alpha(x) = t(x+\alpha)$. With the aid of Theorem 4 it is not difficult to obtain Theorem 5.

Theorem 5. If

$$\widetilde{\Phi}_n(\varphi_r - a_n/n) \geqslant 0, \qquad r = 1, 2, \dots, 2n,$$

then for any $t_n \in \Pi_n$

$$\left\| \int_0^{2\pi} t_n(x+\theta)\Phi_n(\theta)\, d\theta \right\| \leqslant \pi r_n \|t_n\|. \tag{9}$$

Remark. If $\Phi_n(t)$ is defined according to (8) and $t_n(x) = a\cos nx + b \sin nx$, then equality holds in (9).

With the aid of Theorem 5 one can obtain various inequalities for trigonometric polynomials.

Leningrad Institute of Soviet Trade
named after Fr. Engels

Received
25 XI 1964

REFERENCES

[1] P. Erdős, P. Turán, Ann. Math., **41**, 510 (1940).
[2] P. Erdős, P. Turán, Ann. Math., **39**, 703 (1938).
[3] D. L. Berman, DAN, **119**, No. 6 (1958).
[4] M. Riesz, C. R., **158**, 1152 (1914).
[5] S. N. Bernstein, Collected Works, **1**, Publishing House of the Academy of Sciences of the USSR, 1952, p. 257.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.