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Abstract

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MATHEMATICS

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POLYHEDRALLY CLOSED SYSTEMS OF LINEAR INEQUALITIES

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In the present note we consider certain questions connected with extending the following Minkowski theorem to infinite systems of linear inequalities (see ([1])).

The inequality

$$f(x) = b_1 x_1 + \dots + b_{nx} n \leqslant 0$$

is a consequence of the system

$$f_j(x) = a_{j1} x_1 + \dots + a_{jn} x_n \leqslant 0 \qquad (j = 1, 2, \dots, m)$$

(i.e., all its solutions satisfy it) if and only if there exist nonnegative numbers $p_1, \dots, p_m$ such that the identity, with respect to $x \in R^n$,

$$f(x) = \sum_{j=1}^{m} p_j f_j(x).$$

1. Let L be a real linear space and let $f_\alpha(x)$ $(\alpha \in M)$ be real linear (i.e., additive and homogeneous) functions defined on L; here M is some finite or infinite set of indices. The system

$$f_\alpha(x) - a_\alpha \leqslant 0 \qquad (\alpha \in M), \tag{1}$$

where a_α $(\alpha \in M)$ are real numbers, will be called a **system of linear inequalities over** L. For the system (1) with $a_\alpha = 0$ $(\alpha \in M)$, i.e. for a system of the form

$$f_\alpha(x) \leqslant 0 \qquad (\alpha \in M), \tag{2}$$

we introduce the following definitions.

Let L' be some linear space of real linear functions defined on L, containing the elements f_α ($\alpha \in M$), and let C be the convex cone generated by them in L' (the dual cone of the system (2)). The **polyhedral** (L, L')**-closure** $D(C)$ of the cone C will be called the intersection of all sets containing it that are defined in L' by inequalities of the form $x(f) \leqslant 0$ (x is a fixed element of L, and f is an arbitrary element of L'). If such a set is distinct from L', we shall call it an L-half-space of the space L'.

The following obvious generalization of Minkowski' s theorem is valid.

(∗) *In order that the inequality*

$$f(x) \leqslant 0 \qquad (f \in L')$$

be a consequence of the system (2), it is necessary and sufficient that $f \in D(C)$.

In Minkowski' s theorem $D(C) = C$.

Let R be the set of real numbers, $\bar{L} = L \times R$ the linear space of pairs $[x, t]$ ($x \in L,\ t \in R$), $\bar{L}' = L' \times R$ the linear space of pairs $[f, k]$ ($f \in L',\ k \in R$), and $[f, k]([x, t]) = f(x) + kt$. We shall call the system (1) **polyhedrally** (L, L')**-closed** if the dual

the cone of the corresponding system

$$\begin{aligned} f_\alpha(x) - a_\alpha t \leqslant 0 \quad (\alpha \in M), \\ -t \leqslant 0 \end{aligned} \tag{3}$$

coincides with its polyhedral $(\bar{L}, \bar{L}')$-closure.

Remark. It is not difficult to see that a system of the form (2) is polyhedrally (L, L')-closed if and only if its dual cone C coincides with its polyhedral (L, L')-closure $D(C)$. At the same time, the question of the polyhedral (L, L')-closedness of system (1) is completely reduced to the question of the polyhedral $(\bar{L}, \bar{L}')$-closedness of the corresponding system (3).

We shall call a consistent system (1) **finitely** L'**-determined** if every linear inequality

$$f(x) - a \leqslant 0 \quad (f \in L'),$$

which is its consequence (i.e., is not violated by its solutions), is a consequence of some finite subsystem of it.

Taking into account that the inequality $f(x) - a \leqslant 0$ ($f \in L'$) is a consequence of the consistent system (1) if and only if the inequality $f(x) - at \leqslant 0$ is a consequence of system (3), and using further assertion (*), we obtain the following theorem.

Theorem 1. *A consistent system (1) is finitely L'-determined if and only if it is polyhedrally (L, L')-closed.*

In view of the remark to the definition of polyhedral (L, L')-closedness, it follows from this, obviously, that any finite system (1) is polyhedrally (L, L')-closed.

Let L be the finite-dimensional Euclidean space R^n, and let L' be the space conjugate to it. Under the natural identification of the latter with the space $L = R^n$, system (1) takes the form

$$f_\alpha(x) - a_\alpha = (x, x^0_\alpha) - a_\alpha \leqslant 0 \quad (\alpha \in M), \tag{4}$$

where $x^0_\alpha = (a_{\alpha 1}, \dots, a_{\alpha n}) \in L$ ($L = R^n$), and (x, x^0_α) is the scalar product of the elements x and x^0_α. System (3) takes the form

$$\begin{gathered} f_\alpha(x) - a_\alpha t = ([x, t], [x^0_\alpha, -a_\alpha]) \leqslant 0 \quad (\alpha \in M), \\ ([x, t], [\theta, -1]) \leqslant 0, \end{gathered} \tag{5}$$

where $[x, t]$, $[x^0_\alpha, -a_\alpha]$ and $[\theta, -1]$ are elements of the space $\bar{L} = L \times R = R^{n+1}$, and $\theta = (0, \dots, 0) \in L$.

Calling a finitely L'-determined system, in the case under consideration, simply finitely determined, we can now formulate

Corollary 1. *A consistent system (4) is finitely determined if and only if the cone K, generated by the elements $[x^0_\alpha, -a_\alpha]$ $(\alpha \in M)$ and $[\theta, -1]$ in the space $\bar{L}$, is topologically closed in $\bar{L}$.*

Remark 1. The assertion of Corollary 1 is valid in a more general case, when L is a real Hilbert space H, and L' is the space conjugate to it. In this case, in system (4), x^0_α $(\alpha \in M)$ are elements of the space H, and (x, x^0_α) is the scalar product for H. The scalar product in the space $\bar{L} = L \times R = H \times R$ is now introduced by the condition

$$([x, t'], [y, t'']) = (x, y) + t't''$$

((x, y) is the scalar product in H); in this case the space $\bar{L}$ becomes a real Hilbert space.

Remark 2. If the space L' is total on L, i.e., if from $f(x) = 0$ for all f in L' it follows that x is the zero element of L, then the spaces L and L' can be endowed in a natural way with such dual topologies in which both become locally convex linear topological spaces (see, for example, {[2]}, Ch. V,

§ 3). If these topologies are extended in the natural way to the topologies of the linear spaces $\bar{L}$ and $\bar{L}'$ (with preservation of local convexity), then the condition of polyhedral (L, L')-closedness in Theorem 1 passes into the condition of topological closedness of the dual cone of system (3) corresponding to system (1).

As is known, under topological closedness of the infinite set A of vectors $[x^0_\alpha, -a_\alpha]$ $(\alpha \in M)$ in the space R^{n+1} (here $x^0_\alpha \in L = R^n$), the cone K considered in Corollary 1 may turn out to be not topologically closed. Therefore it follows from Corollary 1 that Haar' s assertion (see (3)) on the finite determinacy of system (4) over $\bar{L} = R^n$ with a topologically closed set A in R^{n+1} is false. However, if the set A is not only closed but also bounded, and if, in addition, the common convex hull of the set A and the vector $[0, -1]$ does not contain the zero vector, then the cone K will be topologically closed (see (4), pp. 113 and 110), and therefore the following is true:

Corollary 2 (see (5)). *If the consistent system*

$$f_\alpha(x) - a_\alpha = a_{\alpha 1} x_1 + \ldots + a_{\alpha n} x_n - a_\alpha \leqslant 0 \quad (\alpha \in M) \tag{6}$$

with bounded and closed set $\{(a_{\alpha 1}, \ldots, a_{\alpha n}, -a_\alpha)\}$ $(\alpha \in M)$ *has a solution* $x = (x_1, \ldots, x_n)$ *satisfying the strict inequalities* $f_\alpha(x) - a_\alpha < 0$ $(\alpha \in M)$ *(stable consistency), then it is finitely determined.*

2. It is known that from the consistency and even the stable consistency of all finite subsystems of system (6) its consistency does not follow. In this respect, the following is of some interest.

Theorem 2. *System (6) with a topologically closed cone* K*, generated in the space* R^{n+1} *by the vectors* $(a_{\alpha 1}, \ldots, a_{\alpha n}, -a_\alpha)$ $(\alpha \in M)$ *and* $(0, \ldots, 0, -1)$*, is stably consistent if all its finite subsystems are stably consistent.*

Using Theorem 1, it is not difficult to verify that for arbitrary polyhedrally (L, L')-closed systems of the form (1) the following proposition is valid.

$(**)$ *A polyhedrally* (L, L')*-closed system (1) is consistent if all its finite subsystems are consistent.*

Therefore the following is true.

Theorem $2'$**.** *System (6) with a topologically closed cone* K *is consistent if all its finite subsystems are consistent.*

Remark. Using Corollary 1 of Theorem 1 in the case of systems of the form (4) over a real Hilbert space H (see the remark to Corollary 1 of Theorem 1), it is not difficult to verify that the assertion of Theorem 2′ is valid also in this case. Of course, the cone K is now generated by the elements $[x_\alpha^0, -a_\alpha]$ $(\alpha \in M)$ and $[0, -1]$ of the space $\bar{L} = H \times R$ (see the same remark) and is topologically closed in $\bar{L}$.

3. Let L be a real linear space and L' some nonzero space of real linear functions defined on L. If a convex set Q of the space L is the intersection of some system S of half-spaces (L'-half-spaces) defined by inequalities $f(x) - a \leqslant 0$, where f is a nonzero element of L' and a is a real number, then we shall say that it has a **polyhedral L'-representation** S. If the system S is finite, then the set Q will be called **polyhedral**. An arbitrary system of L'-half-spaces in L will be called **polyhedrally (L, L')-closed** if the system of linear inequalities defining its half-spaces is polyhedrally (L, L')-closed.

We shall call two sets A and B of the space L **completely L'-separable** if it is possible to specify two such planes $f(x)-a=0$ and $f(x)-b=0$ $(a<b,\ f \in L')$ that $f(x) - a \leqslant 0$ for all $x \in A$ and

$f(x) - b \geqslant 0$ for all $x \in B$. In the case where L is the Euclidean space R^n and L' is the space conjugate to L, we shall omit L' in all the terms introduced here (L'-half-space, polyhedral L'-representation, etc.).

Theorem 3. *If two convex sets A and B of the space L, having no common elements, have respectively such polyhedral L'-representations $S(A)$ and $S(B)$ whose union is polyhedrally (L, L')-closed, then they are completely L'-separable.*

This follows with the aid of proposition (**).

Corollary 1 (see [6]). *If two convex polyhedral sets of the space L have no common elements, then they are completely separable, and therefore there exists a plane $f(x) - a = 0$ of the space L strictly separating these sets, i.e., for the elements of one of them $f(x) - a < 0$, and for the elements of the other $f(x) - a > 0$.*

Corollary 2. *If two convex sets A and B of the space R^n, having no common elements, have such polyhedral representations that the outer normals of the half-spaces entering into them generate, as a cone, the whole space R^n, then the sets A and B are completely separable.*

From this proposition there follows, as a consequence, the well-known proposition on the strict separability in the space R^n of two closed convex sets having no common elements, one of which is bounded.

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Note: Figure translations are in progress. See original paper for figures.