ASYMPTOTICS OF SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF THE \(n\)-TH ORDER

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Abstract

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ASYMPTOTICS OF SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF THE n-TH ORDER

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1. Consider the equation

$$ly = \sum_{k=0}^{n} \varepsilon^k q_k(x) y^{(k)} = 0, \qquad q_n(x) \equiv 1, \tag{1}$$

on the interval $[0, \infty)$, where $q_k(x)$ are continuous complex-valued functions of x. Denote by $\lambda_j(x)$ the roots of the equation

$$f(\lambda, x) = \sum_{k=0}^{n} q_k(x) \lambda^k = 0$$

and put

$$\varphi_j(x) = -\frac{\lambda'}{2} \frac{\partial}{\partial \lambda} \left(\ln \frac{\partial f(\lambda, x)}{\partial \lambda} \right) \Bigg|_{\lambda = \lambda_j(x)}, \qquad \tau_0(x) = q_0^{1/n}(x).$$

Let $q_k(x)$ satisfy the conditions:

1) $q_0(x) \neq 0$ for sufficiently large x, the limits

$$\lim_{x \to \infty} q_k \tau_0^{-n+k} = c_k$$

exist and are finite;

2) the equation

$$g(\xi) = \sum_{k=0}^{n} c_k \xi^k = 0 \tag{2}$$

has no multiple roots;

3) $f_{ij}(x) = \mathrm{Re}((\xi_i - \xi_j)\tau_0(x)) \neq 0$ for $i \neq j$ and for sufficiently large x, and

$$\int^{\infty} f_{ij}\,dx = \infty$$

(here ξ_j are the roots of equation (2));

4) the functions $q_k'' q_0^{-2+(2k-1)/n}$, $q_k'' q_0^{-1+(k-1)/n}$ are summable on the interval $[0, \infty)$;

5)

$$q_k' q_0^{-1+k/n} = o\left(\min_{i\neq j} |f_{ij}(x)|\right), \qquad x \to \infty.$$

Theorem 1. Let conditions 1)-5) be satisfied. Then there exists $x_0 > 0$ such that for $x_0 \leq x < \infty$ and $0 < \varepsilon \leq 1$, equation (1) has n linearly independent solutions such that

$$y_j(x) = \exp\left[\int_{x_0}^{x} (\varepsilon^{-1}\lambda_j(t) + \varphi_j(t))\,dt\right](1 + \varepsilon\psi_j(x,\varepsilon)), \tag{3}$$

where $|\psi_j(x,\varepsilon)| < C\varphi(x)$ and $\varphi(x) \to 0$ as $x \to \infty$, C does not depend on ε. Analogous formulas are obtained for $y_j^{(k)}(x)$ for $1 \leq k \leq n$. We also note that $\lambda_j(x) \sim \xi_j \tau_0(x)$ as $x \to \infty$.

2. Consider the equation

$$Ly = \sum_{k=0}^{n} (-1)^k \varepsilon^{2k} \left(p_{n-k}(x) y^{(k)}\right)^{(k)} = 0 \tag{4}$$

On the interval $I = [0, \infty]$, where $p_k(x)$ are continuous complex-valued functions of x and $p_0(x) \neq 0$ for $x \in I$. Let $\Lambda_j(x)$ be the roots of the equation

$$F(\Lambda, x) = \sum_{k=0}^{n} (-1)^k p_{n-k}(x) \Lambda^{2k} = 0.$$

Let the $p_k(x)$ satisfy the conditions:

1) $p_n(x) \neq 0$ for sufficiently large x; the limits

$$\lim_{\infty} p_k p_0^{-1} \tau^{-2k} = c_k$$

exist and are finite, where $\tau(x) = [p_n(x) p_0^{-1}(x)]^{1/2n}$;

2) the equation

$$G(\xi) = \sum_{k=0}^{n} (-1)^k c_{n-k} \xi^{2k} = 0 \tag{5}$$

has no multiple roots;

3) condition 3) of item 1 is fulfilled for the functions $F_{ij}(x) = \mathrm{Re}((\xi_i - \xi_j)\tau(x))$, where ξ_j are the roots of equation (5);

4) the functions

$$p_k'^2 p_0^{-4} \tau^{-4k-1}, \qquad p_k'' p_0^{-2} \tau^{-2k-1}$$

are summable on the interval $[0, \infty)$;

5) $p_k' p_0^{-2} \tau^{-2k-1} = o(\min_{i \neq j} |F_{ij}(x)|)$, $x \to \infty$.

Theorem 2. *Let conditions 1)–5) be fulfilled. Then there exists $x_0 > 0$ such that, for $x_0 \leq x < \infty$ and $0 < \varepsilon \leq 1$, equation (4) has $2n$ linearly independent solutions y_j such that*

$$y_j(x) = [\partial F(\Lambda, x)/\partial \Lambda]^{-1/2}_{\Lambda = \Lambda_j(x)} \times$$

$$\times \exp\left(\varepsilon^{-1} \int_{x_0}^{x} \Lambda_j(t)\, dt \right) (1 + \varepsilon \Psi_j(x, \varepsilon)). \tag{6}$$

The functions Ψ_j have the same properties as the functions ψ_j in Theorem 1.

Analogous formulas hold for the quasi-derivatives $y^{[k]}$ for $1 \leq k \leq 2n-1$. As $x \to \infty$ we have $\Lambda_j(x) \sim \xi_j \tau(x)$.

Formula (6) is new also for binomial equations of the form (4) and coincides with the formulas obtained in (1^{-3}).

3. Let all $p_k(x)$ be real, $\varepsilon = 1$, and let L_0 be the closed symmetric operator generated by the operation L of the form (4) and considered on the interval $[0, \infty)$ (see (2), § 17, item 5). Let the $p_k(x)$ satisfy the conditions:

1′) conditions 1), 2), 4) of item 2 are fulfilled;

2′)

$$\lim_{x \to \infty} p_0(x) = 1, \qquad \lim_{x \to \infty} p_n(x) = \infty;$$

3′)

$$\lim_{x\to\infty} p_k' p_n^{-(2k-1)/2n} = 0.$$

Put $\xi_i' = \xi_i$, if $p_n(x) \to +\infty$ as $x \to \infty$, and $\xi_i' = e^{i\pi/2n}\xi_i$, if $p_n(x) \to -\infty$ as $x \to \infty$.

Theorem 3. *Let $p_k(x)$ satisfy conditions 1′)–3′), and suppose that when $\operatorname{Re}\xi_i' = 0$ and $i \neq j$, either $\operatorname{Re}\xi_i' \neq \operatorname{Re}\xi_j'$, or $\xi_i' = \xi_j'$. Let $\operatorname{Im} G'(\xi_i') \neq 0$ in the latter case. Then:*

1°. *If $\operatorname{Re}\xi_i' \neq 0$ for all i, then the deficiency index of the operator L_0 is equal to (n, n).*

2°. Let $\operatorname{Re}\xi_i' = 0$; $G'(\xi_i') \neq G'(\xi_j')$ for $1 \leq i, j \leq 2k$ and $i \neq j$; $\operatorname{Re}\xi_i' \neq 0$ for the remaining i. Then the defect index of the operator L_0 is equal to $(n+k, n+k)$ or (n, n), depending on whether the integral

$$J = \int^{\infty} p_n - \frac{1}{2n}\, dx$$

converges or diverges.

It is known that the defect index of the operator L_0 is equal to (m, m), where $n \leq m \leq 2n$. An example of an operator L_0 with any possible defect index was first constructed by I. M. Glazman ([4]). For the operators studied in ([2]), $m = n, n+1$, or $2n$. S. A. Orlov ([5]) constructed operators L_0 with any possible defect index; in the case considered by him the equation $Ly = \mu y$ has a regular singular point at $x = +\infty$.

Theorem 3 gives a new broad class of operators L_0 having any possible defect number m, while the point $x = +\infty$ is an irregular singular point for the equation $Ly = \mu y$. Let us also note that for $p_0(x) \equiv 1$ the principal restriction on the order of growth of the function $p_n(x)$ in Theorem 2 is as follows: the integral

$$\int^{\infty} p_n^{-1/2n}\, dx$$

diverges.

Theorem 4. Let L_u be an arbitrary self-adjoint extension of the operator L_0, and suppose that the conditions of Theorem 3 are satisfied. Then in case 1° of Theorem 3 the spectrum of L_u is discrete; in case 2° and $J < \infty$ the spectrum of L_u is discrete and

$$R_\mu = (L_u - \mu E)^{-1}$$

is an integral operator with a Hilbert-Schmidt kernel at all regular points μ; in case 2° and $J = \infty$, the continuous part of the spectrum of L_u fills the entire real axis.

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Note: Figure translations are in progress. See original paper for figures.