ON PERIODIC SOLUTIONS OF NONAUTONOMOUS SYSTEMS

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ON PERIODIC SOLUTIONS OF NONAUTONOMOUS SYSTEMS

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1. In the work ([1]), in considering the question of the branching of solutions of nonlinear equations, we proposed a method for finding the number of all solutions and the form of each solution. It turns out that the method and the considerations used in ([1]) are applicable to the solution of Poincaré' s problem (([7]), Vol. I, Ch. 4) on periodic solutions of a nonautonomous system with an analytic right-hand side. As in ([1]), we arrive at a conclusion about the number of all periodic solutions and the form of each of them. At the same time we establish necessary and sufficient conditions for the convergence of the series obtained by Lyapunov' s method. Hence, as a consequence, it follows that not every formal periodic solution is a genuine one.

2. First consider the equation

$$dx/dt = Ax + \lambda F(t, x, \lambda), \tag{1}$$

where $x = (x_1, x_2, \dots, x_n)$ is the desired vector of the space E_n, $t \in (-\infty, \infty)$, $F = (F_1, F_2, \dots, F_n)$ is a vector-function, continuous and ω-periodic in t, holomorphic jointly in $(x, \lambda) \in G \times \Lambda$ (G is some domain of the space E_n, Λ is a neighborhood of zero in the complex plane), and A is a constant matrix with real elements.

We shall assume that, for the generating equation

$$dy/dt = Ay \tag{2}$$

the resonant case occurs (we do not consider the nonresonant case, since it has been completely studied—see, for example, ([6])) and denote by r ($r \leq n$) the number of its linearly independent ω-periodic solutions, and by R the family of these solutions. Let $\varphi(t) \in R$, and let $x(t, a, \lambda)$ be the solution of the initial-value problem

$$x(0, a, \lambda) = \varphi(0) + a \tag{3}$$

for equation (1). As is known (see, for example, ([5]), p. 56), this solution is representable in the form

$$x(t, a, \lambda) = \varphi(t) + \chi(t, a, \lambda), \tag{4}$$

where χ is a vector whose components are representable in the form of the series

$$\chi_i(t, a, \lambda) = \sum_{m_1 + \dots + m_n + m \geq 1} a^{(i)}_{m_1, \dots, m_n, m}(t)\, \alpha_1^{m_1} \cdots \alpha_n^{m_n} \lambda^m, \tag{5}$$

convergent for $\|a\| \leq \varepsilon$ and $|\lambda| \leq \lambda_0$. The coefficients of the series (5) are determined uniquely by substituting (5) into (1) and solving the resulting recurrent system of linear differential equations with the initial condition (3), which takes the form

$$a^{(i)}_{m_1, \dots, m_n, m}(0) = \begin{cases} 1 & \text{if } m_i = 1 \text{ and } m_k = 0 \ (k \neq i), \\ 0 & \text{for all other values of } m_k. \end{cases}$$

As is known (([5]), p. 168), for ω-periodicity of the solution (4) it is necessary and sufficient that

$$\Phi(\alpha, \lambda) \equiv \chi(\omega, \alpha, \lambda) - \chi(0, \alpha, \lambda) = 0. \tag{6}$$

Every solution $\alpha = \alpha(\lambda)$ of this equation satisfying the condition $\alpha(0) = 0$ and continuous at zero (we shall call such a solution small), when substituted into (4), leads to an ω-periodic solution. Thus, the determination of all ω-periodic solutions of equation (1) branching off from the generating solution $\varphi(t)$ reduces to finding all small solutions $\alpha = \alpha(\lambda)$ of equation (6).

3. Without loss of generality one may assume $\varphi(t) \equiv 0$, since by the substitution $z = y - \varphi(t)$ the general case is reduced to this one.

Let $(y_{ij}(t))$ be the normalized integral matrix of equation (2), and $(\psi_{ij}(t))$ the normalized integral matrix of the adjoint equation. Then (([6]), p. 119) equation (6) takes the form

$$\lambda \int_0^\omega \sum_{\nu=1}^n F_\nu(\tau, x(\tau, \alpha, \lambda), \lambda) \psi_{\nu i}(\tau)\, d\tau = 0, \tag{7}$$

$$\sum_{k=1}^n \alpha_k y_{jk}(\omega) - \alpha_j + \lambda \int_0^\omega \sum_{\nu=1}^n y_{j\nu}(\omega - \tau) F_\nu(\tau, x(\tau, \alpha, \lambda))\, d\tau = 0, \tag{8}$$

where $i = 1, 2, \dots, r$; $j = 1, 2, \dots, n - r$, and the rank of the Jacobian matrix of system (8) with respect to $\alpha_1, \alpha_2, \dots, \alpha_n$ at the point $\alpha_1 = \alpha_2 = \dots = \alpha_n = \lambda = 0$

is equal to $n - r$. Eliminating from (8) $n - r$ unknowns and denoting the remaining unknowns by $\beta_1, \beta_2, \dots, \beta_r$, we obtain, by virtue of (7),

$$g_i(\beta_1, \dots, \beta_r, \lambda) = 0, \qquad i = 1, 2, \dots, r, \tag{9}$$

where the g_i are holomorphic functions in some neighborhood of $\beta_1 = \beta_2 = \dots = \beta_r = \lambda = 0$. If $g_i(0, \dots, 0, 0) \neq 0$ for at least one i, then the generating solution is not continued with respect to λ. Let $\operatorname{rang}(Dg/D\beta)_0 = \rho$. Then, if $\rho = r$, the ([6]) generating solution has a unique ω-periodic continuation in some neighborhood of the point $\lambda = 0$, and in this neighborhood it is an analytic function of λ. The case $\rho = r - 1$ has also been studied ([4]) (see also ([3])). We shall consider the general case.

Lemma 1 ([3]). *If $1 \leq \rho < r$, then the number of small solutions of system (9) coincides with the number of small solutions of the system*

$$\Phi_i(\xi_1, \dots, \xi_{r-\rho}, \lambda) = 0, \qquad i = 1, \dots, p; \quad p = r - \rho, \tag{10}$$

obtained by eliminating from (9) ρ unknowns β_i and replacing the remaining unknowns by ξ_k. Moreover, the order of the series $\Phi_i(\xi_1, \xi_2, \dots, \xi_p, 0)$ is greater than one.

Let us note that system (10) may have more than one small solution, since the order of the series $\Phi_i(\xi_1, \xi_2, \dots, \xi_p, 0)$ is greater than one.

Lemma 2. *The number of ω-periodic solutions of equation (1) tending to $\varphi(t)$ as $\lambda = 0$ coincides with the number of small solutions of system (10).*

It follows from this that (10) is the system of branching equations of the problem under consideration.

4. **Theorem 1.** *Let $p = 1$. Then, if not all coefficients of the branching equation (10) are equal to zero, the number of ω-periodic solutions of the problem under consideration is finite; each of these solutions can be expanded in some neighborhood of $\lambda = 0$ into a convergent series in integral or fractional (with finite common denominator) powers of λ.*

The proof uses propositions from ([2]). In this case the number and form of each solution are determined by the Newton diagram.

Theorem 2. *Let $p = 1$ and let all coefficients of equation (10) be equal to zero. Then the problem under consideration has a family of ω-periodic solutions depending on an arbitrary parameter ξ.*

Let us construct for system (10) ($p > 1$) the pseudopolynomials d_k (1).

Theorem 3. Let $p > 1$. Then, if $d_k \sim 1$ ($k = 1, \dots, p - 1$), the problem under consideration has a finite number of ω-periodic solutions, and each of them can be expanded in some neighborhood of $\lambda = 0$ into convergent series in integral or fractional powers of the parameter λ. If, in addition, $d_p \sim 1$, then the generating solution has no ω-periodic continuations in λ.

Theorem 4. If for some k ($k < p$) one has $d_k \nsim 1$, then the problem under consideration has a finite number of families of ω-periodic solutions, and each such family of solutions depends on $p - k$ arbitrary parameters $\eta_1, \eta_2, \dots, \eta_{p-k}$.

Theorem 5. If all coefficients of system (10) are equal to zero, then the problem under consideration has a family of ω-periodic solutions depending on p arbitrary parameters $\xi_1, \xi_2, \dots, \xi_p$.

5. As is known, Lyapunov' s method consists in seeking the ω-periodic solutions of equation (1) in the form of series in integral or fractional powers of the parameter λ, with ω-periodic coefficients, and the convergence of these series is not assumed in advance (i.e., they are formal series). The coefficients are determined from a recurrent system of differential equations. To prove the convergence of the resulting series, majorants are constructed. Although this method is practically convenient, its implementation is connected with difficulties, since it is not clear in which fractional powers one should seek the solutions and whether the formal solutions found will be actual ones. Moreover, this method does not always make it possible to clarify the question of the number of all solutions.

It is expedient to combine the given method with the method considered in the preceding sections. If the conditions of Theorem 1 or 3 are satisfied, then we know in which powers the solutions should be sought in the form of series. In this case the recurrent process for determining the ω-periodic coefficients will be solvable, and the resulting solutions will be convergent. If the conditions of Theorems 2, 4, or 5 are satisfied, then solutions diverging for every $\lambda \neq 0$ are possible, since the arbitrary parameters can be specialized as functions of λ in such a way that the obtained series diverge. In this case the problem under consideration has an infinite set of ω-periodic solutions, both convergent and divergent (formal); for example, the system

$$dx_1/dt = x_2 + \lambda[(1 - \sin 2t)x_1^2 - (1 + \sin 2t)x_2^2],$$

$$dx_2/dt = -x_1 + \lambda[(1 - \sin 2t)x_1^2 - (1 + \sin 2t)x_2^2]$$

has a family of 2π-periodic solutions of the form

$$x_1 = B(\lambda)(\cos t + \sin t), \qquad x_2 = B(\lambda)(\cos t - \sin t),$$

where $B(\lambda)$ is an arbitrary function of λ.

6. We arrive at a system of branching equations of the form (10) also in the case when the generating equation has the form

$$dy/dt = A(t)y + f(t)$$

with continuous and ω-periodic $A(t)$ and $f(t)$, if this generating equation has r linearly independent ω-periodic solutions.

If the generating equation is not linear, then we pass to the equation in variations ((5), p. 168) and reduce everything to the case considered.

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Note: Figure translations are in progress. See original paper for figures.