UNSTEADY WAVES OVER AN UNDERWATER RIDGE

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Abstract

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R. M. GARIPOV

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In the article, in the linear approximation, the influence of an underwater ridge (for the definition, see below) on the propagation of surface waves arising from an initial disturbance is studied. It is known that the amplitude of waves decreases proportionally to R^{-1}, where R is the distance from the site of the initial disturbance, if the bottom is a horizontal plane. The main conclusion of the article is as follows.

In the presence of an underwater ridge, groups of waves appear that travel over the ridge and whose amplitude decreases as $R^{-\alpha}$, $\alpha = 1/2,\ 1/3,\ 1/4$, depending on the shape of the ridge. $\alpha = 1/4$ if the profile of the ridge satisfies a certain equality. In this case the form of the free surface after the passage of a long interval of time is described by a simple asymptotic formula, which is given below.

The work was carried out under the supervision of M. A. Lavrent' ev. The starting point was the experiments of Sung Tsao (^{1}) and the qualitative results of Munk, Arthur, and others on the linear theory of shallow water. Detailed literature on this question is given in (^{1}).

Let us turn to the equations. Let an ideal incompressible fluid be in a gravitational field, which we assume to be homogeneous, and in equilibrium occupy the domain Ω, bounded by the plane Γ_0 (the free surface) and the surface Γ_1 (the bottom). We place the coordinate plane xy on Γ_0, and direct the z-axis upward. We assume the bottom Γ_1 to be a piecewise-smooth cylindrical surface given by the equation $z = -h(x)$, $h(x) \geq h_0 > 0$. If $h(x) \leq h_\infty$ and outside some finite interval (but not everywhere) $h(x) \equiv h_\infty$, then we shall agree to call such an irregularity of the bottom an underwater ridge. In what follows we put $h_\infty = 1$.

We consider potential motions. The state of the fluid at the time $t = 0$ is regarded as given. In the linear approximation the velocity potential Φ (^{2}) satisfies the equations

$$\begin{gathered} \Omega, \quad \Delta\Phi = 0; \\ \Gamma_0, \quad \Phi_{tt} + \Phi_z = 0; \qquad \Gamma_1, \quad \Phi_n = 0; \\ t = 0, \quad \Gamma_0, \quad \Phi = f_0, \quad \Phi_t = f_1; \end{gathered} \tag{1}$$

the variables are dimensionless. The elevation of the free surface (wave amplitude) is $\zeta = -\Phi_t\big|_{\Gamma_0}$. We seek a solution in the class of functions with finite integral energy, which we denote by $E(\varphi)$; φ is the value of the velocity potential on Γ_0.

Equations (1) can be written in another form, more convenient for the application of general methods. We shall regard φ as the basic unknown function. If φ is known, then the potential Φ can be recovered by solving the boundary-value problem

$$\Omega, \quad \Delta\Phi = 0, \qquad \Phi\big|_{\Gamma_0} = \varphi, \qquad \Phi_n\big|_{\Gamma_1} = 0. \tag{2}$$

For φ we have the operator equation

$$\varphi_{tt} + K\varphi = 0 \tag{3}$$

with initial conditions $\varphi = f_0$, $\varphi_t = f_1$ at $t = 0$. The operator K is defined by the equality $K\varphi = \Phi_z|_{\Gamma_0}$, where Φ is the solution of the boundary-value problem (2).

The study of problem (2) by variational methods (^{3}) shows that the operator K maps $W_2^1(\Gamma_0)$ continuously into $L_2(\Gamma_0)$; it is self-adjoint and positive; it has domain $W_2^1(\Gamma_0)$, if it is regarded as acting in $L_2(\Gamma_0)$. The symmetry and positivity of the operator K formally follow from Green' s formula. The domain of definition of the operator $K^{1/2}$ is $W_2^{1/2}(\Gamma_0)$. Thus equation (3) belongs to the class studied in (^{4}), etc. It has a (generalized) solution if $f_0 \in W_2^{1/2}(\Gamma_0)$, $f_1 \in L_2(\Gamma_0)$, which is unique in the class of functions under consideration. In what follows we assume f_0, f_1 to be such that the necessary smoothness of the solution is ensured.

To the boundary-value problem (2) we apply the Fourier transform with respect to the coordinate y. We obtain

$$\omega, \qquad \Psi_{xx} + \Psi_{zz} - \nu^2\Psi = 0, \qquad \Psi|_{\gamma_0} = \psi, \qquad \Psi_n|_{\gamma_1} = 0, \tag{4}$$

where $\omega, \gamma_0, \gamma_1$ are the intersections of $\Omega, \Gamma_0, \Gamma_1$ with the plane perpendicular to the y-axis. This problem contains the parameter ν instead of the variable y. Define the operator A by the formula $A\psi = \Psi_z|_{\gamma_0}$, where Ψ is the solution of the boundary-value problem (4). The operator A acts in $L_2(\gamma_0)$, and what was said about the operator K is valid for it. In addition, it depends analytically on the parameter ν. In contrast to K, the operator A may have eigenvalues. Its isolated eigenvalues of finite multiplicity will be denoted, with multiplicity taken into account, by λ_k ($k = 1, 2, ...$). The system of corresponding eigenfunctions $\{\psi_k\}$ may be assumed orthonormal. λ_k, ψ_k are defined for $|\nu| \in (\alpha_k, \beta_k)$ and depend analytically on ν (more precisely on ν^2) inside these intervals. The λ_k

actually exist if, for example, the bottom has the form of a submarine ridge. Then they are located in $(0, \nu \operatorname{th} \nu)$ and their number is finite.

Let φ be the solution of equation (3) satisfying the initial conditions. Write

$$\varphi = \sum \varphi_k + \varphi', \tag{5}$$

where

$$\varphi_k = \frac{1}{2\pi} \int_{\alpha_k < |\nu| < \beta_k} e^{-i\nu y} \psi_k(x, \nu) \left(b_k(\nu) \cos \omega_k(\nu) t + a_k(\nu) \frac{\sin \omega_k(\nu) t}{\omega_k(\nu)} \right) d\nu,$$

$$a_k(\nu) = \int e^{i\nu y} \psi_k(x, \nu) f_1(x, y)\, dx\, dy;$$

$b_k(\nu)$ is obtained by replacing f_1 by f_0 in the last equality, and $\omega_k(\nu) = \sqrt{\lambda_k(\nu)} \operatorname{sgn} \nu$.

The meaning of these equalities is revealed by the following

Theorem. All φ_k and φ' satisfy equation (3), and

$$E(\varphi) = \sum E(\varphi_k) + E(\varphi'),$$

and they belong to mutually orthogonal subspaces of the space $L_2(\Gamma_0)$, which are invariant with respect to the operator K.

We do not give the proof here.

Thus the initial data decompose into a sum of mutually orthogonal terms, each of which excites the corresponding component in (5). Therefore, for a special form of the initial conditions, some terms in (5) may be absent. Let us discuss their meaning. Denote $\zeta_k = -\varphi_{kt}$, $\zeta' = -\varphi'_t$.

ζ_k describe groups of waves moving along the y-axis and localized in a certain strip. The order of decay of ζ_k as $|y| \sim t \to \infty$ is determined by the real zeros of the function $\omega''_k(\nu)$ (the prime denotes differentiation). As can be shown by the method of stationary phase (^{2}), in the absence of zeros $\zeta_k \sim t^{-1/2}$; each zero of order m inside (a_k, β_k) gives a contribution $\sim t^{-1/(m+2)}$. If such a zero actually exists, then it is easy to write down the asymptotic form of ζ_k as $t \to \infty$. According to (5), for this it is sufficient to replace $\omega_k(\nu)$, $\psi_k(x, \nu)$, $a_k(\nu)$, $b_k(\nu)$ by the lowest terms of their expansions in a neighborhood of each zero of highest order, set the limits of integration equal to $\pm\infty$, and add the resulting expressions.

Let, for example, $\omega_k''(\nu)$ have in (a_k, β_k) only one zero ν_k of second order. There will be two highest zeros in all: ν_k and $-\nu_k$. Assuming, for simplicity, that $f_0 \equiv 0$ and applying the indicated procedure, we obtain, in the region $y > 0$,

$$\zeta_k = -\frac{\psi_k(x, \nu_k)}{\sqrt[4]{\chi_k t}} \operatorname{Re}\left\{ a_k(\nu_k) e^{-i\nu_k(y - v_k t)} F\left(\frac{y - c_k t}{\sqrt[4]{\chi_k t}}\right)\right\} + O(t^{-1/4}), \qquad (6)$$

where

$$v_k = \omega_k(\nu_k)/\nu_k, \qquad c_k = \omega_k'(\nu_k), \qquad \chi_k = -\omega_k^{\mathrm{IV}}(\nu_k)/6,$$

$$F(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\xi u + u^4/4)}\, du = \frac{1 - i}{4\sqrt{3\pi}}\, |\xi|^{-1/3} e^{\,i^4/|\xi|^{1/3}} + O(|\xi|^{-1}).$$

The function $F(\xi)$ is even and analytic for all ξ.

ζ decays as $t \to \infty$ faster than ζ_k, at least for some nonempty class of initial data.

It follows from the theorem that $\zeta_1, \dots, \zeta'$ are mutually orthogonal at each instant of time t. The asymptotic form of their sum ζ is determined by the most slowly decaying term. If there are several such terms, then the leading terms of their asymptotic expansion cannot cancel one another by virtue of linear independence.

Let us determine when $\omega_k''(\nu)$ have zeros. The functions $\omega_k(\nu)$ are uniquely determined by the form of the bottom, i.e. by $h(x)$. Put $h(x) = 1 - q\varepsilon H(\varepsilon x)$, $q > 0$, $\varepsilon > 0$, and fix H. We find the expansion of ω_k in a neighborhood of $\varepsilon = 0$. In doing so we are interested in the interval of variation of q when zeros exist, and the values of ν near these zeros. Suppose that the indicated values of q, ν as $\varepsilon \to 0$ have orders: $q = O(1)$, $\nu = O(\sqrt{\varepsilon})$. This assumption turns out to be correct.

In the boundary-value problem (4), pass to the variables $\bar{x} = \varepsilon x$, $\bar{\nu} = \nu/\sqrt{\varepsilon}$. Expanding its solution in a series in powers of ε, as is done in shallow-water theory and in (^{6}), we approximate the operator A by the differential operator

$$A\psi = \varepsilon\bar{\nu}^2(1 - \tfrac{1}{3}\varepsilon\bar{\nu}^2 - \varepsilon q H)\psi(\bar{x}) - \varepsilon^2\psi''(\bar{x}) + O(\varepsilon^3). \qquad (7)$$

By $\tilde{\lambda}_k$ we denote the eigenvalues of the differential operator obtained. By definition, $\tilde{\mu}_k = (\tilde{\lambda}_k - \varepsilon\bar{\nu}^2 + \frac{1}{3}\varepsilon^2\bar{\nu}^4)/\varepsilon^2$ are the eigenvalues of the equation

$$\psi''(\bar{x}) + (\tilde{\mu} + \tilde{\nu}^2 H(\bar{x}))\psi(\bar{x}) = 0, \qquad (8)$$

where $\tilde{\nu} = \sqrt{q}\,\bar{\nu}$. With the adopted accuracy,

$$\tilde{\omega}_k'' = \left(\sqrt{\tilde{\lambda}_k}\right)'' = \left(\frac{\varepsilon}{q}\right)^{1/2}\left(-\tilde{\nu} + \frac{q^2}{2}\frac{d^2}{d\tilde{\nu}^2}\left(\frac{\tilde{\mu}_k(\tilde{\nu})}{\tilde{\nu}}\right)\right). \tag{9}$$

Let

$$H(\bar{x}) = 1, \quad |\bar{x}| \le 1; \qquad H(\bar{x}) = 0, \quad |\bar{x}| > 1.$$

Then equation (8) is solved in elementary functions, and it is easy to write $\mu_k(\tilde{\nu})$ explicitly. In this case, as the calculations show, ω_k'' $(k = 1, 2, \ldots)$ for $q > \tilde{q}_k$ has two zeros of the first order, which coalesce at $q = \tilde{q}_k$, forming one zero $\tilde{\nu}_k$ of the second order; for $q < \tilde{q}_k$ there are no zeros. One may expect that, for sufficiently small fixed ε, $\omega_k''(\nu; q, \varepsilon)$ $(k = 1, 2, \ldots, p)$ behaves similarly, i.e., for $q = q_k(\varepsilon)$ it has in (α_k, β_k) a zero $\nu_k(\varepsilon)$ of the second order; for $q (\lessgtr) q_k(\varepsilon)$ it has no zeros (has zeros of the first order); $\tilde{q}_k, \tilde{\nu}_k$ are the limiting values of $q_k(\varepsilon)$, $\sqrt{q_k(\varepsilon)}\, \nu_k(\varepsilon)/\sqrt{\varepsilon}$ as $\varepsilon \to 0$. For the surfaces Γ_1, which were defined above as an underwater ridge, such a dependence of ω_k'' on q, with H, ε fixed, seems to us typical. Table 1 gives the first terms of the expansion in a neighborhood of $\varepsilon = 0$ of the quantities entering the asymptotic formula (6), for $k = 1, 2, 3$.

Table 1

	$k = 1$	2	3		$k = 1$	2	3
q_k	21.5	16.8	17.2	$(1 - c_k)/\varepsilon$	11.6	10.3	11.7
$\nu_k/\sqrt{\varepsilon}$	0.69	1.11	1.45	$\varkappa_k\sqrt{\varepsilon}$	3	12	16
$(1 - \nu_k)/\varepsilon$	9.9	6.0	5.2				

It is interesting to note that in the expansion (7) of the operator A we had to retain one term more (the second in the parentheses) than is customary in the linear theory of shallow water, in order to capture the indicated dependence of ω_k'' on q. In the shallow-water approximation in (9) the term $-\tilde{\nu}$ in the parentheses is absent, and the presence or absence of zeros of ω_k'' does not depend on q. For example, in the case of the considered form of $H(\bar{x})$, there is in $(0, \infty)$ only one zero of the first order for all $q > 0$.

Institute of Hydromechanics
Siberian Branch
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.