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Abstract

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V. S. BREZHNEV

ON THE QUESTION OF THE MOTION OF A CHARGED PARTICLE IN THE THEORY OF GRAVITATION

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It is known that in Einstein' s theory of gravitation (1) the motion of a material particle is completely determined by the properties of the gravitational field, and the equations of motion are the integrability conditions of the field equations (2,3).

The situation is somewhat different when one considers the motion of a charged particle in interacting gravitational and electromagnetic fields. The interaction of the gravitational field with the electromagnetic field is usually taken into account (2) by introducing into the right-hand side of Einstein' s gravitational equations the energy-momentum tensor of the electromagnetic field. The latter is chosen from the requirement that the conservation law for the total energy-momentum tensor of the sources of the gravitational field should lead to the equations of motion of a charged particle containing the Lorentz force (2).

A more consistent formulation is that of the equations of the gravitational-electromagnetic field without using the equations of motion of the particle, which must follow from the compatibility conditions of the field equations themselves.

The Einstein-Maxwell equations in vacuum, as is known (4), can be derived from the principle of least action

$$\delta \int_{\Omega_4} \left(R - \frac{\varkappa}{8\pi} F_{\alpha\beta} F^{\alpha\beta} \right) \sqrt{-g}\, d^4x = 0 \tag{1}$$

and have the form

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - {}^1/_2 R g_{\alpha\beta} - \frac{\varkappa}{4\pi} \left(-F_{\alpha\lambda} F^{\lambda}_{\beta} + {}^1/_4 F_{\mu\nu} F^{\mu\nu} g_{\alpha\beta} \right) = 0, \tag{2}$$

$$E^{\mu} \equiv -\frac{\varkappa}{2\pi} F^{\mu\nu}{}_{;\nu} = 0 \quad \left(\varkappa = \frac{8\pi k}{c^4} \right). \tag{3}$$

Here $R_{\alpha\beta}$ and R are the contracted curvature tensor and the scalar curvature of the four-dimensional pseudo-Riemannian space, $F_{\alpha\beta} = \varphi_{\beta,\alpha} - \varphi_{\alpha,\beta}$ is the electromagnetic-field tensor; φ_α is the electromagnetic potential; $G_{\alpha\beta}$, E^μ denote the variational derivatives of the Lagrange function with respect to the corresponding variables.

For what follows it is necessary to note that the variational equation (1) is invariant both with respect to the group of arbitrary coordinate transformations

$$x'^\alpha = f^\alpha(x^\beta), \tag{4}$$

and with respect to the group of gauge transformations

$$\varphi'_\alpha = \varphi_\alpha + \lambda_{,\alpha} \tag{5}$$

of the electromagnetic potential.

Since the Einstein-Maxwell equations (2), (3) are also invariant with respect to both of the above-mentioned groups of transformations, these equations cannot be completely independent of one another and must satisfy

correspond to 5 differential identities. To obtain these identities, let us represent equation (1) in the form

$$\int_{\Omega_4} (G_{\alpha\beta}\delta g^{\alpha\beta} + E^\alpha \delta\varphi_\alpha)\sqrt{-g}\, d^4x = 0 \tag{6}$$

and we shall assume that the variations $\delta g^{\alpha\beta}$, $\delta\varphi_\alpha$ are caused by an infinitesimal coordinate transformation and an infinitesimal gauge transformation.

Performing an infinitesimal coordinate transformation

$$x'^\alpha = x^\alpha + \delta x^\alpha \tag{7}$$

with the gauge fixed, we have [5]

$$\delta g^{\alpha\beta} = g^{\alpha\lambda}\delta x^\beta{}_{,\lambda} + g^{\lambda\beta}\delta x^\alpha{}_{,\lambda} - g^{\alpha\beta}{}_{,\lambda}\delta x^\lambda; \tag{8}$$

$$\delta\varphi_\mu = -\varphi_\lambda \delta x^\lambda{}_{,\mu} - \varphi_{\mu,\lambda}\delta x^\lambda. \tag{9}$$

Substituting these values into equation (6) and integrating by parts under the condition that δx^λ vanishes on the boundary of the region Ω_4, we obtain

$$G_\lambda{}^\mu{}_{;\mu} + \frac{1}{2}F_{\lambda\mu}E^\mu - \frac{1}{2}E^\mu{}_{;\mu}\varphi_\lambda \equiv 0. \tag{10}$$

In the case of an infinitesimal gauge transformation

$$\delta\varphi_\mu = \delta\lambda_{,\mu} \tag{11}$$

with the coordinate system fixed, we arrive at the differential identity

$$E^\mu{}_{;\mu} \equiv 0, \tag{12}$$

taking which into account, identity (10) can be given the form

$$G^{\lambda\mu}{}_{;\mu} + \frac{1}{2} F^\lambda{}_\mu E^\mu \equiv 0. \tag{13}$$

Turning now to the Einstein–Maxwell equations with sources

$$G^{\alpha\beta} = -\varkappa T^{\alpha\beta}, \tag{14}$$

$$E^\mu = -2\varkappa s^\mu, \tag{15}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor and s^μ is the current vector, we conclude, taking (13), (12) into account, that equations (14), (15) are compatible only when

$$T^{\lambda\mu}{}_{;\mu} + F^\lambda{}_\mu s^\mu = 0, \tag{16}$$

$$s^\mu{}_{;\mu} = 0. \tag{17}$$

If the gravitational-electromagnetic field is produced by a flow of charged material particles, then the energy-momentum tensor and the current vector can be represented [4,5] in the form

$$T^{\alpha\beta} = \mu c^2 u^\alpha u^\beta, \tag{18}$$

$$s^\alpha = \rho u^\alpha, \tag{19}$$

where μ is the mass density, ρ is the charge density, and u^α is the four-velocity field of the flow. In this case, from (16), (17) we obtain the equations of motion of the particles

$$\mu u^\lambda{}_{;\nu} u^\nu + \frac{\rho}{c^2} F^\lambda{}_\mu u^\mu = 0, \tag{20}$$

the continuity equation

$$(\mu u^{\alpha})_{;\alpha} = 0 \tag{21}$$

and the law of conservation of charge

$$(\rho u^{\alpha})_{;\alpha} = 0. \tag{22}$$

In the case of a test particle having mass m and charge e, we may set in equations (20)

$$a^{\alpha} = d\xi^{\alpha}/ds, \tag{23}$$

$$\mu = m\delta_3(\vec{x} - \vec{\xi}), \tag{24}$$

$$\rho = e\delta_3(\vec{x} - \vec{\xi}), \tag{25}$$

which gives

$$\frac{d^2\xi^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\lambda\mu}\frac{d\xi^{\lambda}}{ds}\frac{d\xi^{\mu}}{ds} + \frac{e}{mc^2}F^{\alpha}{}_{\beta}\frac{d\xi^{\beta}}{ds} = 0. \tag{26}$$

Thus, it has been proved that the motion of charged material particles is completely determined by the properties of the gravitational-electromagnetic field, while the equations of motion of the particles producing the field are the integrability conditions of the field equations.

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Note: Figure translations are in progress. See original paper for figures.