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Abstract

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ON AN INTEGRABLE CASE OF THE RIEMANN BOUNDARY-VALUE PROBLEM FOR SEVERAL FUNCTIONS

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Suppose that it is required to determine n piecewise-analytic functions $\varphi_i(z)$ $(i = 1, \dots, n)$ with line of jumps $L + M$, whose boundary values on the simple smooth contour $L + M$ satisfy the conditions

$$\varphi_i^+(t) = g_{ik}(t)\varphi_k^-(t) + f_i(t) \qquad (t \in L + M),$$

$$\det \|g_{ik}(t)\| \neq 0 \qquad (i, k = 1, \dots, n). \tag{1}$$

Here $g_{ik} = 0$ for $i \neq k$ on M, while on L in the matrix $\|g_{ik}\|$ all entries except $g_{12}, g_{23}, \dots, g_{n-1,n}, g_{n1}$ are equal to zero. The functions $g_{ik}(t)$ and $f_i(t)$ are assumed to be piecewise continuous, satisfying a Hölder condition on intervals of continuity.

In the general case the Riemann boundary-value problem for many functions reduces to a system of Fredholm equations. The most complete investigations of this problem are presented in the monograph of N. I. Muskhelishvili ([1]), in the book of N. P. Vekua ([2]), and in the survey article of F. D. Gakhov ([3]). In ([3]) two cases of integrability in closed form of the Riemann boundary-value problem for several functions are also indicated: 1) the matrix $\|g_{ik}(t)\|$ is functionally commutative; 2) $\|g_{ik}(t)\|$ is a product of two matrices whose elements are functions analytic, respectively, in the interior and exterior domains of the contour $L + M$, except for a certain number of points at which poles may occur. The latter case is a generalization of cases found earlier by N. P. Vekua and D. A. Kveseleva ([4,5]) and by N. P. Vekua ([6]). In the present note the problem posed above is solved in closed form.

1°. Introduce new piecewise-analytic functions $\Phi_i(z)$ (index $i = 1, \dots, n$) by the formulas

$$\Phi_i(z) = \frac{\varphi_i(z)}{V_i(z)} - \frac{1}{2\pi i}\int_M \frac{f_i(t)\,dt}{V_i^+(t)(t-z)}. \tag{2}$$

Here $V_i(z)$ is the canonical solution of the Riemann boundary-value problem ([1],[7])

$$\varphi_i^+(t) = g_{ii}(t)\varphi_i^-(t) \qquad (t \in M) \tag{3}$$

(where it is assumed that $\varphi_i(z)$ is analytic everywhere outside the contour M).

The functions $\Phi_i(z)$ thus defined, as is easy to verify, have a discontinuity only on the line L, while on the contour M they are continuous.

Passing in the boundary condition (1) to the functions $\Phi_i(z)$, we obtain the following boundary-value problem:

$$\Phi_i^+(t) = G_{ik}(t)\Phi_k^-(t) + F_i(t) \qquad (t \in L) \qquad (i, k = 1, \dots, n). \tag{4}$$

Here

$$G_{12}(t) = \frac{V_2(t)}{V_1(t)} g_{12}(t), \qquad G_{23}(t) = \frac{V_3(t)}{V_2(t)} g_{23}(t), \dots$$

$$\dots, G_{n-1,n}(t) = \frac{V_n(t)}{V_{n-1}(t)} g_{n-1,n}(t), \qquad G_{n1}(t) = \frac{V_1(t)}{V_n(t)} g_{n1}(t); \tag{5}$$

$$F_i(t) = \frac{1}{V_i(t)} \left\{ f_i(t) + g_{i,i+1}(t) \frac{V_{i+1}(t)}{2\pi i} \int_M \frac{f_{i+1}(\tau)\, d\tau}{V_{i+1}^+(\tau)(\tau - t)} \right\} - \frac{1}{2\pi i} \int_M \frac{f_i(\tau)\, d\tau}{V_i^+(\tau)(\tau - t)}$$

(index $i = 1, 2, \dots, n-1$);

$$F_n(t) = \frac{1}{V_n(t)} \left\{ f_n(t) + g_{n1}(t) \frac{V_1(t)}{2\pi i} \int_M \frac{f_1(\tau)\, d\tau}{V_1^+(\tau)(\tau - t)} \right\} - \frac{1}{2\pi i} \int_M \frac{f_n(\tau)\, d\tau}{V_n^+(\tau)(\tau - t)}.$$

All G_{ik}, except $G_{12}, G_{23}, \dots, G_{n-1,n}, G_{n1}$, are equal to zero.

For a contour L that divides the complex z-plane into its interior and exterior domains, the boundary-value problem (4), (5) (as also the more general problem (4) with a triangular matrix) is solved by a simple renaming of the functions.

2°. Let the contour L consist of some number m of simple open smooth curves not passing through the infinitely distant point. The functions $\varphi_i(z)$ are sought in the class of analytic functions having no more than polynomial growth at infinity and integrable at the ends of the curves and at the points of discontinuity of the coefficients $g_{ik}(t)$.

Definition. By a **canonical solution** of the boundary-value problem (4), (5) for an open contour L not passing through the infinitely distant point, we shall mean piecewise analytic functions $X_i(z)$ $(i = 1, \dots, n)$, satisfying the boundary conditions on the contour L

$$X_i^+(t) = G_{ik}(t) X_k^-(t) \qquad (i, k = 1, \dots, n) \tag{6}$$

and possessing the following properties:

1) The function $X_i(z)$ has ν zeros at the points $z = c_{ij}$ $(j = 1, \dots, \nu;\ i = 1, \dots, n)$, and at the remaining points of the finite part of the plane has zero order, with the exception, possibly, of the points of discontinuity of the coefficients $G_{ik}(t)$ and the ends of the curves L_k—the points $z = a_k$ and $z = b_k$ $(L = L_1 + \dots + L_n)$. The numbers ν and c_{ij} are such that the finite system consisting of $(m-1)(n-1)$ equations

$$\int_L \tau^i \frac{q_1(\tau) + \Delta^{j(n-1)} q_2(\tau) + \Delta^{j(n-2)} q_3(\tau) + \dots + \Delta^j q_n(\tau)}{B_{mj}^+(\tau)} \, d\tau = 0 \tag{7}$$

$$(i = 0, 1, 2, \dots, m-2; \quad j = 1, 2, \dots, n-1),$$

is consistent. Here $B_{mj}^+(\tau)$ denotes the limiting value on the left bank of the cut (traversed in the direction from a_k to b_k) of the function analytic outside L

$$B_{mj}(z) = \prod_{k=1}^{m} (z - a_k)^{(n-j)/n} (z - b_k)^{j/n}. \tag{8}$$

Moreover, in formula (7) the following notation is used:

$$q_i(\tau) = \ln \left\{ G_{i,i+1}(\tau) \prod_{j=1}^{\nu} \frac{\tau - c_{i+1,j}}{\tau - c_{ij}} \right\} \qquad (i = 1, 2, \dots, n-1),$$

$$q_n(\tau) = \ln \left\{ G_{n1}(\tau) \prod_{j=1}^{\nu} \frac{\tau - c_{1j}}{\tau - c_{nj}} \right\}, \qquad \Delta = \exp \frac{2\pi\sqrt{-1}}{n}. \tag{9}$$

2) At the ends of the curves $z = a_k$ and $z = b_k$ and at the points of discontinuity of the coefficients $g_{ik}(t)$, the class of the function $X_i(z)$ coincides with the prescribed class of the function $\Phi_i(z)$ $(i = 1, \dots, n)$.

3) At infinity the function $X_i(z)$ has the highest possible order.

Let us find the canonical solution. Introduce new functions $\chi_i(z)$, having zero order in the whole z-plane and almost bounded at the endpoints of the curves L_k:

$$\chi_i(z) = \ln\left\{X_i(z)\prod_{j=1}^{\nu}(z-c_{ij})^{-1}\right\} \quad (i=1,\dots,n). \tag{10}$$

Taking logarithms, from the system of conditions (6) we obtain the boundary-value problem for $\chi_i(z)$

$$\begin{aligned} &\chi_i^+(t) = \chi_{i+1}^-(t) + q_i(t), \qquad i = 1,\dots,n-1,\\ &\chi_n^+(t) = \chi_1^-(t) + q_n(t). \end{aligned} \tag{11}$$

It is not difficult to show that the solution of the Riemann boundary-value problem (11), having the highest possible order at infinity and belonging to the class of functions almost bounded at the endpoints of the curves L_k, has the form

$$\chi = \Lambda^{-1}\Gamma. \tag{12}$$

Here

$$\chi = \begin{Vmatrix} \chi_1(z)\\ \vdots \\ \chi_n(z)\end{Vmatrix}, \qquad \Gamma = \begin{Vmatrix} \Gamma_1(z)\\ \vdots \\ \Gamma_n(z)\end{Vmatrix}, \qquad \Lambda = \begin{Vmatrix} 1 & \Delta^{n-1} & \Delta^{n-2} & \cdots & \Delta\\ 1 & \Delta^{n-2} & \Delta^{n-4} & \cdots & \Delta^2\\ 1 & \Delta^{n-3} & \Delta^{n-6} & \cdots & \Delta^3\\ . & . & . & \cdots & .\\ 1 & \Delta & \Delta^2 & \cdots & \Delta^{n-1}\\ 1 & 1 & 1 & \cdots & 1 \end{Vmatrix},$$

$$\Gamma_n(z) = \frac{1}{2\pi i}\int_L \frac{q_1(\tau)+\cdots+q_n(\tau)}{\tau - z}\,d\tau, \tag{13}$$

$$\Gamma_j(z) = \frac{B_{mj}(z)}{2\pi i}\int_L \frac{q_1(\tau)+\Delta^{j(n-1)}q_2(\tau)+\Delta^{j(n-2)}q_3(\tau)+\cdots+\Delta^{j}q_n(\tau)}{B_{mj}^+(\tau)(\tau-z)}\,d\tau$$

$$(j = 1,2,\dots,n-1).$$

Finally, it is convenient to write the canonical solution in the form

$$X_i(z) = \prod_{j=1}^{\nu}(z-c_{ij})\prod_{k=1}^{m}(z-t_k)^{-\varkappa_k}e^{\chi_i(z)} \quad (i=1,\dots,n). \tag{14}$$

Here the arguments of the functions $g_{12}(t), g_{23}(t), \dots, g_{n1}(t)$ on the curve L_k are chosen in such a way that, under continuous variation of the arguments from the points $z = a_k$ and $z = b_k$ and from an arbitrarily chosen singular point $z = t_k$ on L_k, the singularities of the functions $X_i(z)$ at the discontinuity points of the coefficients $g_{12}, \dots, g_{n1}$ and at the points $z = a_k$ and $z = b_k$ coincide with the prescribed class $\varphi_i(z)$, while at the point $z = t_k$ the prescribed class is obtained by a choice of the integer $\varkappa_k$ (quite analogously to the Riemann boundary-value problem for one function (7)).

The solvability of the system of equations (7) and the existence of the canonical solution (14) are easily proved if the contour L is completed by arbitrary curves to a closed contour, if one recalls the fact that the Riemann problem with a piecewise-continuous matrix of coefficients can be reduced to the case of a continuous matrix ([3]), and if one uses the existence of a solution of the homogeneous Riemann problem with a continuous matrix in the class of analytic functions having no more than polynomial growth at infinity ([1,2]).

It should be noted that for $m = 1$ (and also for constant G_{ik}) the solution of problem (4) is substantially simplified, since one may take $\nu = 0$, and the conditions (7), together with the constants c_{ij}, disappear.

3°. Introducing the functions $\psi_i(z)$

$$\psi_i(z) = \frac{\varphi_i(z)}{X_i(z)} - \sum_{j=1}^{\nu} \frac{a_{ij}}{z - c_{ij}} \quad (i = 1, \dots, n), \tag{15}$$

where

$$a_{ij} = \lim_{z \to c_{ij}} \frac{(z - c_{ij})\varphi_i(z)}{X_i(z)},$$

we write the nonhomogeneous problem (4) in the form

$$\psi_i^+(t) = \psi_{i+1}^-(t) + h_i(t) \quad (i = 1, \dots, n-1),$$

$$\psi_n^+(t) = \psi_1^-(t) + h_n(t) \quad (t \in L). \tag{16}$$

Here

$$h_i(t) = \frac{F_i(t)}{X_i^+(t)} + \sum_{j=1}^{\nu} \left(\frac{a_{ij}}{t - c_{ij}} - \frac{a_{i+1,j}}{t - c_{i+1,j}} \right) \quad (i = 1, \dots, n-1),$$

$$h_n(t) = \frac{F_n(t)}{X_n^+(t)} + \sum_{j=1}^{\nu} \left(\frac{a_{nj}}{t - c_{nj}} - \frac{a_{1j}}{t - c_{1j}} \right).$$

The solution of problem (16) differs from the solution of problem (11) only by the possibility that $X_i(z)$ vanishes at some endpoints of the curves L_k, and by the related possibility of changing the class of the functions $\psi_i(z)$ at these endpoints of the curves. We shall not give the corresponding formulas.

4°. Let now the contour L pass through the infinitely distant point. This case is reduced to the preceding one by means of a fractional-linear transformation. One may also apply a direct solution, entirely analogous to the one considered, taking as the special point some finite point of the z-plane not lying on the contour L.

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Note: Figure translations are in progress. See original paper for figures.