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Abstract

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MATHEMATICS

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AN APPROXIMATION THEOREM AND STRUCTURAL ISOMORPHISMS

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Every isomorphism between groups gives rise to a structural isomorphism (projection) between them. Known examples ([1]) of structurally isomorphic groups show that the converse assertion is not valid in general. The question of the conditions under which a projection is induced by a group isomorphism is reduced, with the aid of a local theorem ([2], [3]), to the corresponding problem for finitely generated groups. Here an approximation theorem on projections is proved, in a certain sense dual to the local one.

1. A system $\mathfrak{M} = \{H_\alpha \mid \alpha \in I\}$ of normal divisors H_α (where α ranges over some set I) of a group G will be called a **filter** in G if: 1) the intersection of any two members of the filter H_α and H_β contains some $H_\gamma \in \mathfrak{M}$; 2) the intersection of all $H_\alpha \in \mathfrak{M}$ coincides with the identity e of the group G. We note that condition 2 is equivalent to saying that for every element a of G there is a member of the filter not containing a. Strengthening 2), we shall call a filter **strict** if for every $a \in G$ there is an $H_\alpha \in \mathfrak{M}$ not containing any nonzero power of this element. Clearly, in a torsion-free group every filter consisting of its isolated normal divisors is strict. In a periodic group every filter is strict.

2. A group G is called **approximable by Θ-groups** (groups belonging to the class Θ) if in G there is a filter $\mathfrak{M} = \{H_\alpha \mid \alpha \in I\}$ such that for every $H_\alpha \in \mathfrak{M}$ the quotient group G/H_α belongs to the class Θ. Thus, for example, a group is (purely) nilpotently approximable if the class Θ consists of all (purely) nilpotent groups. It is known ([8]) that the class of nilpotently approximable groups coincides with the class of groups in which the lengths of the descending central series do not exceed ω, the first limit ordinal.

3. Let us record the properties of a filter that we shall need: 1) let H_α be an arbitrary member of $\mathfrak{M}$; then the collection of all H_β from $\mathfrak{M}$ lying in H_α also forms a filter; 2) for any finite set of elements of G there is a member of the filter containing none of them.

4. If a projection is induced by exactly one isomorphism, then it is induced by exactly one anti-isomorphism. Therefore, in what follows we shall speak

only of corresponding isomorphisms.

Theorem 1 (approximation). *Let Θ be a class of groups such that every projection of a Θ-group is induced by exactly one isomorphism. Suppose, moreover, that the group G is approximable by Θ-groups with a strict filter $\mathfrak{M} = \{H_\alpha \mid \alpha \in I\}$. Then every projection $\varphi(G) = G^\varphi$ of the group G onto some group G^φ, preserving normality of the members of the filter, is induced by exactly one isomorphism between G and G^φ.*

Proof. Denote by φ_α the projection induced by the projection φ of the factor group G/H_α onto $G^\varphi/H_\alpha^\varphi$. By hypothesis, φ_α is induced by exactly one isomorphism ψ_α. There is a definite relation among all $\psi_\alpha (\alpha \in I)$; we shall establish it. Consider ψ_α and ψ_β under the assumption that $H_\beta \subset H_\alpha$. In this case ψ_β induces a certain isomorphism $\psi_\beta^{(\alpha)}$ between G/H_α and $G^\varphi/H_\alpha^\varphi$, namely: if

$$\psi_\beta(aH_\beta) = b'H_\beta^\varphi, \qquad b' \in G^\varphi, \tag{1}$$

then we set

$$\psi_\beta^{(\alpha)}(aH_\alpha) = b'H_\alpha^\varphi. \tag{2}$$

It is easy to see that the notation (2) does not depend on the choice of representatives in adjacent classes and that $\psi_\beta^{(\alpha)}$ is indeed an isomorphism. It is also obvious that $\psi_\beta^{(\alpha)}$ induces the projection φ_α, and therefore, by uniqueness, $\psi_\beta^{(\alpha)} = \psi_\alpha$. Thus we have the following relation: if (1) holds, then

$$\psi_\alpha(aH_\alpha) = b'H_\alpha^\varphi. \tag{3}$$

We now define a one-to-one correspondence ψ between the elements of G and G^φ. Let a be any element of G. Find in $\mathfrak{M}$ an H_α which contains no nonzero power of a. Consider the cyclic subgroup $\{a\} = A$. Suppose that $\varphi(A) = A^\varphi = \{a'\}$. Then $A^\varphi \cap H_\alpha^\varphi = e$. Let $\psi_\alpha(aH_\alpha) = b'H_\alpha^\varphi$, where $b'H_\alpha^\varphi \in G^\varphi/H_\alpha^\varphi$. We verify that the class $b'H_\alpha^\varphi$ contains one and only one power ${a'}^k$ of the element a'. Indeed, the isomorphism ψ_α induces the projection φ_α, under which $\varphi_\alpha(\{aH_\alpha\}) = \{b'H_\alpha^\varphi\}$. But $\varphi_\alpha(\{aH_\alpha\}) = \{A^\varphi, H_\alpha^\varphi\}/H_\alpha^\varphi$; consequently, $\psi_\alpha(aH_\alpha) \in \{A^\varphi, H_\alpha^\varphi\}/H_\alpha^\varphi$. Each element on the right-hand side of the last inclusion has the form ${a'}^k H_\alpha^\varphi$. Hence, for some k_α,

$$\psi_\alpha(aH_\alpha) = {a'}^{k_\alpha} H_\alpha^\varphi. \tag{4}$$

The exponent k_α (for the given a) is determined here uniquely (modulo the order, if the order of a is finite), since $A^\varphi \cap H_\alpha^\varphi = e$. The element a' itself does not depend on α, for it is taken from the cyclic group A^φ, which is determined by the projection φ. A priori the exponent in (4) may be regarded as depending on α. In fact, k_α does not depend on α. Indeed, choose in $\mathfrak{M}$ two distinct members H_α and H_β and consider two possibilities.

1) $H_\beta \subset H_\alpha$. From the preceding it follows, similarly to (4), that

$$\psi_\beta(aH_\beta) = a'^{\,k_\beta} H_\beta^\varphi. \tag{5}$$

Hence, using the relation expressed by equality (3), we have $\psi_\alpha(aH_\alpha) = a'^{\,k_\beta} H_\alpha^\varphi$. Consequently $a'^{\,k_\beta} H_\alpha^\varphi = a'^{\,k_\alpha} H_\alpha^\varphi$. This, however, in view of the choice of H_α, is possible only when $k_\alpha = k_\beta$.

2) $H_\beta \not\subset H_\alpha$. Suppose, as before, that (4) and (5) hold. Find in $\mathfrak{M}$ an H_γ which lies in $H_\alpha \cap H_\beta$. It is clear that $A \cap H_\gamma = e$. For H_γ there also exists an exponent k_γ for which $\psi_\gamma(aH_\gamma) = a'^{\,k_\gamma} H_\gamma^\varphi$. But now, by 1), $k_\gamma = k_\alpha$ and $k_\gamma = k_\beta$. Consequently, the exponent $k = k_\alpha = k_\beta$ does not depend on α, i.e. it is determined uniquely by the element a and by the generator a' of the group A^φ. As for the element a'^k, it depends only on a. We now define the desired correspondence ψ by the equality

$$\psi(a) = a'^k. \tag{6}$$

It was shown above that ψ is a one-to-one correspondence. Let us verify that ψ is a mapping of G onto the whole group G^φ. Choose any $a' \in G^\varphi$. Find H_α^φ not containing a'. Then the isomorphism ψ_α^{-1} indicates that element $a \in G$ which corresponds to a'. Hence the correspondence ψ is one-to-one. Further, if $H_\alpha \cap A = e$, then by the definition of ψ (see (6)),

$$\psi_\alpha(aH_\alpha) = \psi(a) H_\alpha^\varphi. \tag{7}$$

On the other hand, using relation (3), we immediately obtain that the same relation (7) is valid also for all H_α (and not only for those for which $H_\alpha \cap A = e$). It remains to verify that ψ is an isomorphism. To this end choose in G an arbitrary pair of elements a and b. Let H_α be an arbitrary member of $\mathfrak{M}$. Then from (7) we have $\psi(ab)H_\alpha^\varphi = \psi_\alpha(abH_\alpha)$. But ψ_α is an isomorphism; therefore

$$\psi_\alpha(abH_\alpha) = \psi_\alpha(aH_\alpha)\psi_\alpha(bH_\alpha).$$

But, according to (7),

$$\psi_\alpha(abH_\alpha) = \psi(a)H_\alpha^\varphi \psi(b) H_\alpha^\varphi = \psi(a)\psi(b)H_\alpha^\varphi.$$

Hence it follows that, for every α,

$$\psi^{-1}(b)\psi^{-1}(a)\psi(ab) \in H_\alpha^\varphi.$$

This means, taking account of the property of the members of the filter, that

$$\psi(ab) = \psi(a)\psi(b).$$

Thus ψ is an isomorphism and, by its very construction, induces the projection φ. It remains to note that if there exist two isomorphisms ψ and χ inducing the same φ, then for some H_α they both give rise to isomorphisms ψ_α and χ_α inducing the projection φ_α, which contradicts the assumed uniqueness of the isomorphism inducing φ_α.

5. A projection $\varphi(G) = G^\varphi$ is called **normal** (**weakly normal**) if the subgroup $H^\varphi = \varphi(H)$ is invariant (and isolated) in G^φ if and only if H is a normal (and isolated) divisor in G.
6. It is known (5) that, under every projection $\varphi(G) = G^\varphi$ of a torsion-free group G, to every isolated locally nilpotent normal divisor H of it there corresponds in G^φ likewise an isolated locally nilpotent normal divisor H^φ. Taking this fact into account, the following is valid.

Theorem 2. Let the group G possess an ascending isolated invariant series

$$e = G_0 \subset G_1 \subset \dots \subset G_\alpha \subset \dots \subset G_\beta = G$$

with locally nilpotent factors $G_{\alpha+1}/G_\alpha$ (β an arbitrary ordinal number). Then every projection $\varphi(G) = G^\varphi$ of the group G is weakly normal.

Proof. Consider in G an arbitrary isolated normal divisor H. Construct in H the invariant series

$$e = H_0 \subseteq H_1 \subseteq \dots \subseteq H_\alpha \subseteq \dots \subset H_\beta = H,$$

in which $H_\alpha = H \cap G_\alpha$ ($\alpha = 1, \dots, \beta$). By assumption, G_1, and hence also H_1, is an isolated locally nilpotent normal divisor in G. Therefore the subgroup $H_1^\varphi = \varphi(H)$ has in G^φ the same properties. Suppose the isolation and invariance of H_γ^φ have been proved for all $\gamma < \alpha$. For a limit α the invariance and isolation of H_α^φ are evident. For a nonlimit α it has already been proved that $H_{\alpha-1}^\varphi$ is invariant and isolated in G^φ. The projection φ gives rise to a projection of the factor group $G/H_{\alpha-1}$ onto $G^\varphi/H_{\alpha-1}^\varphi$, under which to the isolated locally nilpotent normal divisor $H_\alpha/H_{\alpha-1}$ there corresponds the same kind of normal divisor $H_\alpha^\varphi/H_{\alpha-1}^\varphi$ in $G^\varphi/H_{\alpha-1}^\varphi$. Consequently, the subgroup H_α^φ is normal and isolated in G^φ. On the basis of the induction carried out, one may regard the subgroup H^φ as invariant and isolated in G^φ.

7. Let Θ be a class of torsion-free groups such that every projection of a Θ-group is induced by exactly one isomorphism. Then from the approximation theorem and Theorem 2 we immediately obtain

Theorem 3. If the group G possesses an ascending isolated invariant series with locally nilpotent factors and is approximated by Θ-groups, then each of its projections is induced by exactly one isomorphism.

Applying Theorem 3 to the case where Θ is the class of pure nilpotent groups, we obtain a generalization of the result of paper [6]:

Corollary 1. *Suppose that in a group G there is a decreasing central series of length ω with pure factors, and also an increasing isolated invariant series with locally nilpotent factors. Then every projection of G arises under exactly one isomorphism.*

If one now uses the local theorem on projections, then Corollary 1 (and the main theorem from [6]) can be generalized. The combined use of the local and approximation theorems makes it possible to encompass ever broader classes of groups.

8. Let us indicate one more application of Theorem 1. In [7] it is proved that every free polynilpotent group (for the definition see [4]) is determined by its structure. This result also contains

Theorem 4. *Every projection of a free polynilpotent (in particular, free soluble of arbitrary class n) group arises under exactly one of its isomorphisms.*

Proof. A free polynilpotent group is approximated by pure nilpotent groups [7]. At the same time, from [6] it is known that pure nilpotent (nonabelian) groups lie in the class Θ. On the other hand, the group G has a finite soluble isolated invariant series. (Therefore every projection of the group G is weakly normal; Theorem 2.) It remains to apply Theorem 3 and thereby complete the proof.

9. Theorem 1 can be strengthened if the notion of a Θ-class is broadened by including in it every group each projection of which is induced by no more than a finite number of isomorphisms. In this case, the existence of at least one isomorphism inducing the projection $\varphi(G) = G^{\varphi}$ is established with the aid of the notion of an inverse spectrum, similarly to the way it is done in [3] in the proof of the local theorem on projections.

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