MATHEMATICS

B. V. POPOV

1965

SovietRxiv

Abstract

Full Text

MATHEMATICS

B. V. POPOV

SEMIGROUPS OF TRANSFORMATIONS OF RANK NOT EXCEEDING TWO

(Presented by Academician A. I. Mal' tsev on 13 XI 1964)

The **rank of a transformation** a of a set Ω is the cardinality of the set $a(\Omega)$. In semigroups of transformations, the rank of a transformation is one of the most important characteristics. All questions connected with semigroups of transformations of rank 1 are solved trivially. The next step may be the study of semigroups of transformations whose rank does not exceed two. The fact ([1]) that an arbitrary quasiorder relation (with the exception of trivial ones) is exhaustively characterized by the semigroup of all its endomorphisms of rank not exceeding two indicates, on the one hand, that semigroups of transformations of rank not exceeding two may be quite complex, and, on the other hand, that the study of such semigroups is of definite interest.

In the present note we consider the class of semigroups $\mathfrak{A}$, defined as follows. A semigroup A of full transformations of a set Ω of rank not exceeding two belongs to the class $\mathfrak{A}$ if it contains at least one transformation of rank 1 and

$$\bigcup_{a\in A} a(\Omega) = \Omega.$$

We note that semigroups consisting only of transformations of rank 2 belong to the class of completely simple semigroups without zero, which is one of the most studied.

In the note we use generally accepted logical symbols. The meaning of terms from the theory of semigroups and binary relations that are used without special references may be found in ([2], [3]). The necessary notions concerning representations of semigroups are contained in ([4]). Finally, $|X|$ will everywhere denote the cardinality of the set X.

1. Let A be an arbitrary semigroup whose set of left zeros A_0 is nonempty. Define in A the equivalence ε_0 as the equivalence closure of the relation $\rho \subset A\times A$

$$(a,b)\in\rho \leftrightarrow ((|aA_0|\neq 1 \;\&\; aA_0 = bA_0)\vee(a\in bA\setminus A_0))$$

and the equivalence θ

$$(a,b)\in\theta\leftrightarrow(\forall_{x\in A_0}(ax=bx)\ \&\ \forall_{x\in A}(ax\in A_0\leftrightarrow bx\in A_0)).$$

Denote by $I(\mathfrak{A})$ the class of all semigroups isomorphic to semigroups of the class $\mathfrak{A}$.

Theorem 1. *A semigroup A will be a semigroup of the class $I(\mathfrak{A})$ if and only if:*

$$A_0=\emptyset, \tag{1}$$

$$abc\in A_0\to(ab\in A_0\vee bc\in A_0), \tag{2}$$

$$(ab\notin A_0\ \&\ |abA_0|=1)\to|bA_0|=1, \tag{3}$$

$$(\forall_{x\in A}(ax\in A_0)\ \&\ |aA_0|=1)\to a\in A_0, \tag{4}$$

$$|\varepsilon_0(a)A_0|\leq 2, \tag{5}$$

$$\varepsilon_0\cap\theta=\Delta, \tag{6}$$

whatever $a,b,c\in A$ may be.

2. The descriptions of representations of semigroups available in the literature are either too general ([4]-[6]) and are nonconstructive in character, or else describe representations of a special form ([7]-[9]). The method given below for obtaining all isomorphic representations of an arbitrary semigroup $A\in I(\mathfrak{A})$ in the class $\mathfrak{A}$ cannot be derived from the results of the above-mentioned works.

3. Let A be an arbitrary semigroup from $I(\mathfrak{A})$. The totality A_1 of elements $a\in A$ such that $|\varepsilon_0(a)A_0|=1$ is a subsemigroup of the semigroup A. For any pair of elements $a,b\in A_1$ we define the set $A_{a,b}: x\in A_{a,b}\leftrightarrow xaA_0=xbA_0$. We next define on A_1 the equivalence ε_1

$$a,b)\in\varepsilon_1\leftrightarrow(\forall_{x\in A_{a,b}}(xa\in A_0\leftrightarrow xb\in A_0)\ \&\ \forall_{x\notin A_{a,b}}(xa\in A_0\leftrightarrow xb\notin A_0)).$$

(There is the inclusion $(\varepsilon_0)_{A_1}\subset(\varepsilon_1)_{A_1\setminus A_0}\cup\Delta_{A_1}$.)

4. Let ε be an arbitrary equivalence on A_1 such that

$$(\varepsilon_0)_{A_1} \subset \varepsilon \subset (\varepsilon_1)_{A_1 \setminus A_0} \cup \Delta_{A_1}, \qquad \varepsilon \cap \theta_{A_1} = \Delta_{A_1}.$$

These conditions are satisfied, for example, by $(\varepsilon_0)_{A_1}$. We agree to identify, in the factor set A_1/ε, an equivalence class consisting of only one element $a \in A_1$ with this element. Let

$$(\varepsilon_1(a) \cap A_0) \setminus (\varepsilon_0 \circ \theta)(\varepsilon(a))A_0 = C_{\bar{a}},$$

where $\bar{a}$ is the equivalence class under ε containing the element $a \in A_1$. Specify an arbitrary mapping f of the set A_1/ε into itself satisfying the conditions

$$f(\bar{a}) \in C_{\bar{a}} \cup \bar{a}, \qquad a \in A_0 \to f(\bar{a}) = \bar{a}, \qquad (a, b \notin A_0 \;\&\; f(\bar{a}) = f(\bar{b})) \to \bar{a} = \bar{b}.$$

The identity mapping f_0 satisfies these conditions. Put $\bar{A} = f(A_1/\varepsilon)$. Obviously, $A_0 \subset \bar{A}$.

5. Define a mapping $P_{\varepsilon,f}$ of the semigroup A into the semigroup of all full transformations of the set $\bar{A}$, putting, for arbitrary $a \in A$ and $\bar{x} \in \bar{A}$,

$$P_{\varepsilon,f}(a)(\bar{x}) = \begin{cases} ax, & \text{if } ax \in A_0, \\ f(\bar{a}), & \text{if } ax \notin A_0 \text{ and } a \in A_1, \\ \varepsilon_0(a)A_0 \setminus axA_0, & \text{if } ax \notin A_0 \text{ and } a \notin A_1. \end{cases}$$

Let us note that $\varepsilon_0(a)A_0 \setminus axA_0$ is a well-defined element of A_0 (p. 1, (5)) and that $P_{\varepsilon,f}(a)(\bar{x})$ does not depend on the choice of the representative x in the class $\bar{x}$.

6. **Theorem 2.** Let $A \in I(\mathfrak{A})$. The mapping $P_{\varepsilon,f}$, defined in item 5, is an isomorphic representation of the semigroup A in the class $\mathfrak{A}$. Every isomorphic representation of the semigroup A in the class $\mathfrak{A}$ differs only inessentially from some representation of the type $P_{\varepsilon,f}$. Two representations $P_{\varepsilon,f}$ and $P_{\varepsilon',f'}$ of the semigroup A differ inessentially from one another if and only if $\varepsilon = \varepsilon'$ and $f = f'$.

As follows from item 4, an arbitrary semigroup A of the class $I(\mathfrak{A})$ always has a representation $P_1 = P_{\varepsilon,f}$, where $\varepsilon = (\varepsilon_0)_{A_1}$ and $f = f_0$. Sometimes this representation turns out to be unique.

7. **Theorem 3.** The representation P_1 (item 6) of a semigroup $A \in I(\mathfrak{A})$ will be the unique (up to inessential difference) isomorphic representation of the semigroup A in the class $\mathfrak{A}$ if and only if

$$\varepsilon_1(a) \subset (\varepsilon_0 \circ \theta \circ \varepsilon_0)(a) \cup (\varepsilon_0 \circ \theta \circ \varepsilon_0)(a)A_0,$$

whatever $a \in A_1 \setminus A_0$ may be.

It may happen that $A_1 = A_0$; then P_1 will be a representation by left shifts on the ideal A_0 and, moreover, the unique isomorphic representation of the semigroup A in the class $\mathfrak{A}$. The validity of the latter assertion also follows from the results of the work (10).

8. Denote by $\mathfrak{A}_0$ the subclass of the class $\mathfrak{A}$ consisting of semigroups satisfying the condition
$$\alpha \neq \beta \to \exists_{a \in A}(a(\alpha) \neq a(\beta)),$$
where α, β are arbitrary transformed symbols.

Semigroups of the class $\mathfrak{A}_0$ are d-subsemigroups (11) of the semigroup of all transformations. The reasons why d-subsemigroups should be considered deserving of attention are set forth in (11).

Theorem 4. *A semigroup A is a semigroup of class $I(\mathfrak{A}_0)$ if and only if it has properties (1)–(5) of Sec. 1 and*

$$(\varepsilon_0 \cup \varepsilon_1) \cap \theta = \Delta, \tag{7}$$

$$(\varepsilon_1)_{A_0} = \Delta_{A_0}, \tag{8}$$

$$(\varepsilon_0 \circ \theta)(\varepsilon_1(a) \setminus A_0) \cap \varepsilon_1(a) = \emptyset \quad \text{for all } a \in A_1 \setminus A_0. \tag{9}$$

9. Let $A \in I(\mathfrak{A}_0)$. From (7) it follows that $|\varepsilon_1(a) \cap A_0| \leqslant 1$ for all $a \in A_1$. Put
$$\varepsilon^* = (\varepsilon_1)_{A_1 \setminus A_0} \cup \Delta_{A_1}.$$
Define a mapping f^* of the set A_1/ε^* into itself by setting, for each $\bar{x}^* \in A_1/\varepsilon^*$, $f^*(\bar{x}^*) = \varepsilon_1(a) \cap A_0$, if $\varepsilon_1(a) \cap A_0 \neq \emptyset$, and $f^*(\bar{x}^*) = \bar{x}^*$ otherwise. By (7) and (9), ε^* and f^* satisfy the conditions of Sec. 4. The representation $P_0 = P_{\varepsilon^*, f^*}$ is a representation of the semigroup A in the class $\mathfrak{A}_0$. Moreover:

Theorem 5. *Let $A \in I(\mathfrak{A}_0)$. Every isomorphic representation of the semigroup A in the class $\mathfrak{A}_0$ differs inessentially from the representation P_0.*

10. Let P' and P be representations of an arbitrary semigroup A by transformations of the sets Ω' and Ω. We shall write $P' \leqslant P$ if there exists a mapping π of the set Ω onto the set Ω' such that

$$\forall_{a \in A} \, \forall_{\alpha \in \Omega}(P'(a)(\pi(\alpha)) = \pi(P(a)(\alpha))).$$

In the terminology of (5), the mapping π will be a representative homomorphism of the transformation semigroup $P(A)$. In the case $P' \leqslant P$ we shall say that the representation P' is obtained from the representation P by reduction, borrowing this term from [12]. We shall also say that the mapping π effects the reduction of the representation P to the representation P'.

Theorem 6. *$P_{\varepsilon',f'} \leqslant P_{\varepsilon,f} \leftrightarrow (\varepsilon \subset \varepsilon' \ \& \ \forall_{\bar{x}\in A_1/\varepsilon} (f(\bar{x}) = \bar{x} \vee f(\bar{x}) = f'(\bar{x}')))$, where $\bar{x} \in A_1/\varepsilon$, $\bar{x}' \in A_1/\varepsilon'$. Let $P_{\varepsilon',f'} \leqslant P_{\varepsilon,f}$. The mapping $\pi : \pi(\bar{x}) = f'(\bar{x}')$, and only it, effects the reduction of the representation $P_{\varepsilon,f}$ to the representation $P_{\varepsilon',f'}$.*

11. Denote by $\mathfrak{R}(A)$ the set of all representations of the semigroup $A \in I(\mathfrak{A})$ of the type $P_{\varepsilon,f}$ with the relation $\leqslant$ defined in Sec. 10. It is easy to establish that $\mathfrak{R}(A)$ is an ordered set.

Theorem 7. *Let $A \in I(\mathfrak{A})$. The ordered set $\mathfrak{R}(A)$ is a complete upper semi-lattice, and P_1 (Sec. 6) is its greatest element. If $A \in I(\mathfrak{A}_0)$, then $\mathfrak{R}(A)$ is a complete lattice, in which P_0 (Sec. 9) is the least element.*

We indicate the sup of an arbitrary set $(P_{\varepsilon_i,f_i})_{i\in I}$ of representations from $\mathfrak{R}(A)$, and, in the case $A \in I(\mathfrak{A}_0)$, also the inf. We shall denote by $\bar{x}^i$ the equivalence class modulo ε_i containing the element $x \in A_1$, considered as an element of the set A_1/ε_i.

$$\sup(P_{\varepsilon_i,f_i})_{i\in I} = P_{\varepsilon,f},$$

where

$$\varepsilon = \bigcap_{i\in I} \varepsilon_i,$$

and f maps the set A_1/ε into itself in the following way: if

$$\exists_{a\in A_0} \forall_{i\in I} (f_i(\bar{x}^i) = a),$$

then $f(\bar{x}) = a$; in all other cases $f(\bar{x}) = \bar{x}$.

Let $A \in I(\mathfrak{A}_0)$. $\inf(P_{\varepsilon_i,f_i})_{i\in I} = P_{\varepsilon,f}$, where ε is the transitive closure of the relation μ

$$(x,y) \in \mu \leftrightarrow \left((x,y \in \bigcup_{i\in I} \varepsilon_i) \vee \exists_{i,k\in I} (\bar{x}^i \neq f_i(\bar{x}^i) = f_k(\bar{y}^k) \neq \bar{y}^k) \right),$$

and the mapping f is defined as follows: if

$$\exists_{i\in I} (f_i(\bar{x}^i) \neq \bar{x}^i),$$

then $f(\bar{x}) = \varepsilon_1(a) \cap A_0$; in all other cases $f(\bar{x}) = \bar{x}$.

12. We shall say that a representation P of a semigroup A by transformations of a set Ω is the most economical among some collection of representations of the semigroup A, if for every representation of the semigroup A from this collection by transformations of a set Ω' the inequality $|\Omega| \leq |\Omega'|$ holds.

As a consequence of Theorem, item 11, we obtain:

Whatever the set of isomorphic representations of the semigroup $A \in I(\mathfrak{A})$ in the class $\mathfrak{A}$ may be, there exists the most economical isomorphic representation of the semigroup A in the class $\mathfrak{A}$, from which one can obtain, by reduction (item 10), every representation of this set; if the semigroup A is finite, then a representation with this property is unique (up to an inessential distinction). In particular, every isomorphic representation of the semigroup A in the class $\mathfrak{A}$ can be obtained from the representation P_1 (item 6) by reduction, and Theorem 6 and item 4 make it possible to indicate the form of the mapping π that effects this reduction.

Finally, if $A \in I(\mathfrak{A}_0)$, then the representation P_0 is the most economical among the isomorphic representations of the semigroup A in the class $\mathfrak{A}$. In the case of a finite semigroup A, every isomorphic representation of the semigroup A with this property differs inessentially from the representation P_0.

I express my deep gratitude to E. S. Lyapin for useful advice and attention to the work.

Leningrad State
Pedagogical Institute
named after A. I. Herzen

Received
4 XI 1964

REFERENCES

[1] L. M. Gluskin, Uspekhi Mat. Nauk, 16, no. 5, 157 (1961).
[2] E. S. Lyapin, *Semigroups*, Moscow, 1960.
[3] J. Riguet, *Cybernetics Collection*, vol. 7, IL, 1963.
[4] E. S. Lyapin, Mat. sbornik, 52, no. 1, 589 (1960).
[5] V. V. Wagner, Mat. sbornik, 38, no. 2, 203 (1956).
[6] B. M. Schein, Mat. sbornik, 55, no. 4, 279 (1961).
[7] E. J. Tully, Am. J. Math., 83, no. 3, 533 (1961).
[8] B. M. Schein, Uspekhi Mat. Nauk, 18, no. 3, 215 (1963).
[9] I. S. Ponizovskii, Sibirsk. matem. zhurn., 5, no. 4, 896 (1964).
[10] E. S. Lyapin, *Collection in Memory of N. G. Chebotarev*, Kazan Univ. Press, 1964.
[11] L. M. Gluskin, Mat. sbornik, 47, no. 1, 111 (1959).
[12] R. Stoll, Duke Math. J., 11, 251 (1944).