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Abstract

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ON THE INFLUENCE OF INERTIA ON THE DEPOSITION OF AEROSOL PARTICLES FROM A FLOW AT SUBCRITICAL STOKES NUMBERS

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The study of processes of washout of aerosol particles from the atmosphere encounters a number of difficulties associated with the need to take simultaneous account of inertial and diffusion effects in a certain range of sizes. If, at Stokes numbers $k > k_{\text{cr}}$, investigators simply added the action of both effects, assuming that the two mechanisms of deposition act independently, then at $k < k_{\text{cr}}$ this cannot be done, since inertial capture is impossible. Nevertheless, from physical considerations it is clear that displacement of particles under the action of inertial forces from the trajectories of fluid elements also occurs at $k < k_{\text{cr}}$, and therefore, in accordance with the change in the trajectories of particle motion, the conditions of diffusional deposition also change.

A system of equations for the simultaneous description of diffusion and inertial phenomena was first obtained in work ([1]) and has the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\tau}(\mathbf{v} - \mathbf{u}) + \frac{\mathbf{F}(\mathbf{r}, t)}{m}; \tag{1}$$

$$\frac{\partial n}{\partial t} = -\operatorname{div}(n\mathbf{v}) + D\Delta n; \tag{2}$$

$\mathbf{u}$ is the velocity of the medium; $\mathbf{F}$ is an external force; $\mathbf{v}$, n, D, m, τ are, respectively, the velocity, concentration, diffusion coefficient, mass, and relaxation time of the particles*.

Investigation of the system (1), (2) is hampered by the nonlinearity of equation (1). Therefore, analysis of the system carried out for some flow model that permits an analytic solution of the problem becomes of great importance. As such a model one may choose the case of hyperbolic flow, which permits an analytic solution of equation (1). This flow corresponds to flow past an infinite wall ($x = 0$) and, what is especially important, approximates real flows around symmetric bodies near the forward critical point ([2]). Since for it the velocity field (in the half-plane $x \leq 0$) can be represented as

$$u_x = -ax, \qquad u_y = by,$$

then, in dimensionless form, for the stationary case the system (1), (2) can be written as follows:

$$k\frac{dv_x}{dt} + v_x + ax = 0,$$

$$k\frac{dv_y}{dt} + v_y - by = 0, \tag{3}$$

$$n \operatorname{div} \mathbf{v} + \mathbf{v} \operatorname{grad} n = \frac{1}{\mathrm{Pe}}\Delta n,$$

where $k = u_\infty \tau / L$ is the Stokes number, $\mathrm{Pe} = u_\infty L/D$ is the Peclet number, and L is the characteristic dimension.

* We note that in some cases, when the change in the external force or in the velocity field over the length of the free path cannot be neglected, instead of D there appears an effective diffusion coefficient ([1]).

The first two equations have been investigated in detail in [2]. For $k < k_{\mathrm{cr}} = 1/4a$, near the surface one may put

$$v_x = \lambda x, \qquad v_y = \mu y, \tag{4}$$

where

$$\lambda = \left(-1 + \sqrt{1 - 4ak}\right)/2k, \qquad \mu = \left(-1 + \sqrt{1 + 4bk}\right)/2k.$$

Then system (3) reduces to an ordinary differential equation reducible to Whittaker' s equation:

$$\frac{1}{\mathrm{Pe}}\frac{d^2 n}{dx^2} - \lambda x \frac{dn}{dx} - (\lambda + \mu) n = 0. \tag{5}$$

The solution of (5) is expressed in terms of degenerate hypergeometric functions $\Phi(\alpha, \gamma, |\lambda| \mathrm{Pe}\, x^2/2)$, and the dimensionless flux density to the obstacle j, using the asymptotic expansion of $\Phi(\alpha, \gamma, {}^1/_2 |\lambda| \mathrm{Pe}\, x^2)$, can be represented in the form*

$$j = \Gamma(1-\delta)(|\lambda|/a)^{1/2}(|\lambda|\mathrm{Pe}/2x_0^2)^{\delta} j_{\mathrm{diff}}. \tag{6}$$

Here $\Gamma(1-\delta)$ is the gamma function; $\delta = |\lambda+\mu|/2|\lambda|$; x_0 is the initial coordinate; $j_{\mathrm{diff}} = (2a/\pi\mathrm{Pe})^{1/2}$ is the flux density obtained from the solution of the convective-diffusion equation, i.e., without allowance for inertial effects.

Analysis shows that the change in the flux compared with j_{diff} is due to two processes—the motion of particles along trajectories lying closer to the wall (the approach effect) and the growth of the number concentration of particles along a trajectory because the divergence of $\mathbf{v}$ is not zero [3]. The first effect corresponds to the factor $(|\lambda|/a)^{1/2}$, which for $0 \le k \le k_{\mathrm{cr}}$ has limits of variation $1 \div \sqrt{2}$; the second effect can give a more substantial change. For $k \approx k_{\mathrm{cr}}$ (bearing in mind that in dimensionless variables $x_0 \sim 1$, and $\delta \approx 0.3$) the factor can be very large, since in many practically important problems $\mathrm{Pe} \gg 1$. Thus, for this case one may assume that

$$j = j_{\mathrm{diff}}(a\mathrm{Pe})^{0.3}. \tag{7}$$

Thus, allowance for inertia substantially changes the magnitude of the flux to an obstacle (especially for $\mathrm{Pe} \gg 1$) and changes the dependence of j on the number Pe: $j_{\mathrm{diff}} \sim \mathrm{Pe}^{-0.5}$, $j \sim \mathrm{Pe}^{-0.2}$. This change in the flux is mainly due to the growth of the number concentration along particle trajectories.

Determination of the fluxes of aerosol particles to bodies of a specific configuration is complicated by the need to solve equation (1); however, in some cases approximate methods of solution may be used. It can be shown that near a surface, where the effect of diffusion is manifested, the equations of motion admit simplification and can be investigated within the framework of the qualitative theory of Emden equations [3]. For $k < k_{\mathrm{cr}}$, as shown in [2], the particle velocity asymptotically adapts to the flow velocity. Proceeding from this, one can solve the problem in the approximation of a two-layer model, calculating the change in concentration outside the boundary layer and solving the diffusion equation within it. In this case, if $\mathrm{Pe} \gg 1$, then the capture coefficient is small, and the change in concentration can be calculated only for the region near the line of symmetry from which particle capture occurs. This problem was investigated in detail in [2]. The diffusive part of the problem may be considered as follows.

* The flux density j is equivalent to the local capture coefficient [2].

For example, in the case of a sphere, the equations of particle motion (1) in the region $y = (r-1) \ll 1$ reduce to the form

$$kv_y\partial v_y/\partial y - kv_\theta\partial v_y/\partial\theta - kv_\theta^2 + v_y = -{}^3/2\, y^2\cos\theta,$$
$$kv_y\partial v_\theta/\partial y + kv_\theta\partial v_\theta/\partial\theta + kv_{yv_\theta} + v_\theta = {}^3/2\, y\sin\theta. \tag{8}$$

Estimating the order of the terms entering system (8), under the assumption that $v_y \sim y^2$, $v_\theta \sim y$ (in the diffusion boundary layer the particle velocity adjusts to the flow velocity), and discarding terms of the next order of smallness, it is not difficult to show that

$$v_y = \left({}^9/4k\sin^2\theta - {}^3/2\cos\theta\right) y^2, \qquad v_\theta = {}^3/2\, y\sin\theta,$$
$$\operatorname{div}\mathbf{v} = 3ky\sin^2\theta. \tag{9}$$

Let us proceed to consideration of the diffusion equation. For the narrow region near the surface of the sphere it admits the simplifications proposed by Levich ([4]) in the treatment of convective diffusion. Then our diffusion equation takes the form

$$v_y\frac{\partial n}{\partial y} + v_\theta\frac{\partial n}{\partial\theta} + {}^9/2\,kyn\sin^2\theta = \frac{1}{\mathrm{Pe}}\frac{\partial^2 n}{\partial y^2}. \tag{10}$$

The first integral of the equations of motion in the boundary layer has the form

$$\psi = y\sin\theta e^{-3/2k\cos\theta}. \tag{11}$$

Passing in equation (10) from the variables θ, y to the variables θ, ψ, we shall have:

$$\frac{\partial n}{\partial\theta} + 3kn\sin\theta = \frac{2}{3\mathrm{Pe}}\sin^2\theta e^{{}^9/2\,k\cos\theta}\frac{1}{\psi}\frac{\partial^2 n}{\partial\psi^2}. \tag{12}$$

Further, making the substitution $n = ce^{3k\cos\theta}$ and introducing the variable

$$\varphi = \frac{2}{3\mathrm{Pe}}\int \sin^2\theta e^{{}^9/2\,k\cos\theta}\,d\theta,$$

we shall have

$$\frac{\partial c}{\partial y} = \frac{1}{\psi}\frac{\partial^2 C}{\partial\psi^2}. \tag{13}$$

We seek the solution of (13) in the form $C = C(\eta)$, where $\eta = \psi^3/\varphi$. In this case equation (13) reduces to the solution of an ordinary differential equation of second order.

As a result of integration and transition to the original variables we obtain:

$$n = A_1 e^{3k\cos\theta}\int_0^\eta x^{-2/3}e^{-x/9}\,dx + A_2;$$

$$\eta = y^3 \sin^3 \theta e^{9/2 \cos\theta} \Bigg/ \left(\frac{2}{3\mathrm{Pe}} \int_0^\theta \sin^2 \beta e^{9/2 k \cos\theta} \, d\beta + A_3 \right). \tag{14}$$

Let us find the constants of integration. From the condition that $n = 0$ at $y = 0$, it follows that $A_2 = 0$. From the condition that n does not become identically zero at $\theta = 0$, it follows that $A_3 = 0$. From the condition $n = n_1$ as $\psi \to \infty$,

$$A_1 = n_1 e^{-3k} \Bigg/ \int_0^\infty x^{-2/3} e^{-x/9} \, dx.$$

Thus, finally we obtain:

$$n = n_1 e^{-3k(1-\cos\theta)} \int_0^\eta x^{-2/3} e^{-x/9} \, dx \Bigg/ \int_0^\infty x^{-2/3} e^{-x/9} \, dx. \tag{15}$$

Hence it is not difficult to obtain an expression for the flux density:

$$j = \left(6^{2/3} e^{-3k} / 2\Gamma\left({}^1/3\right) \mathrm{Pe}^{2/3}\right) \sin\theta e^{9/2 k \cos\theta} \Bigg/ \left[\int_0^\theta \sin^2 \beta e^{9/2 k \cos\beta} \, d\beta \right]^{1/3}. \tag{16}$$

The total dimensionless flux I (equivalent to the capture coefficient)

$$I = 2\pi \int_0^\pi j \sin\theta \, d\theta = \frac{6^{2/3} \pi n_1(\mathrm{k}) e^{-3\mathrm{k}}}{\Gamma(1/3) \mathrm{Pe}^{2/3}} \int_0^\pi \frac{\sin^2 \theta \, e^{\theta/2 \, \mathrm{k} \cos\theta} \, d\theta}{\left[\int_0^{-\theta} \sin^2 \beta \, e^{\theta/2 \, \mathrm{k} \cos\beta} \, d\beta \right]^{1/3}} \tag{17}$$

or

$$I = \frac{6^{1/3} \pi^{5/3}}{\Gamma(1/3)} \frac{e^{-3\mathrm{k}}}{\mathrm{k}^{2/3}} I_1^{2/3} \left(\frac{\theta}{2} \mathrm{k} \right) \frac{n_1(\mathrm{k})}{\mathrm{Pe}^{2/3}}. \tag{18}$$

For small k, expanding the Bessel function I_1 in a series, we obtain

$$I = \frac{9\pi^{5/3}}{2\sqrt[3]{3}\,\Gamma(1/3)} \left(1 - 3\mathrm{k} + \frac{99}{16}\mathrm{k}^2 - \frac{9}{2}\mathrm{k}^3 + \cdots \right) n_1(\mathrm{k}) \, \mathrm{Pe}^{-2/3}. \tag{19}$$

For $\mathrm{k} = 0$, expression (19) becomes the flux for convective diffusion to a sphere, in good agreement with Levich' s solution ([4]), since the coefficient in (19) is equal to 7.848. For $\mathrm{k} \neq 0$, the flux does not coincide with the case of convective diffusion. This difference is due to the centrifugal force in the boundary layer, which decreases the flux, and to some increase in the calculated concentration because div **v** is not equal to zero. Outside the boundary layer the accumulation of concentration is very substantial, and it determines the difference between n_1 and n_0. Table 1 gives the values of $n_1(\mathrm{k})^*$, $g(\mathrm{k}) = \mathrm{k}^{-2/3} e^{-3\mathrm{k}} I_1^{2/3}(\theta/2\,\mathrm{k})$, and $f(\mathrm{k}) = n_1 g$. The latter function characterizes the change in the capture coefficient caused by the influence of inertia.

Table 1

k	n_1	g	f
0.2	1.85	1.01	1.86
0.4	3.01	0.667	2.01
0.6	5.34	0.485	2.59
0.8	10.6	0.378	4.00
1.0	32.0	0.308	9.86

It is seen from the table, for example, that if the large drops have $R = 15\,\mu$, and the small ones $r = 6.6\,\mu$ ($\mathrm{Pe} = 2.2 \cdot 10^5,\ \mathrm{k} = 1$), then inertial effects increase the capture coefficient from 0.0022 to 0.022.

An analogous calculation was carried out for the case of flow around a cylinder. The flux I then has the form

$$I = \frac{6^{2/3} 3^{1/2} \pi^{1/3} \Gamma^{2/3}(3/4)}{\Gamma(1/3)} \frac{\omega^{1/6} e^{-4\mathrm{k}\omega}}{\mathrm{k}^{1/6}} I_{1/4}^{2/3}(6\omega\mathrm{k}) \frac{n_1(\mathrm{k})}{\mathrm{Pe}^{2/3}}, \tag{20}$$

where $\omega = 2/(1 - 2\ln\gamma\,\mathrm{Re}/4)$; $\gamma = 1.7811$.

For small k, equation (20) takes the form:

$$I = 4.59\,(1 - 4\omega\mathrm{k} + 12.8\omega^2\mathrm{k}^2 + \cdots)\,\omega^{1/3} n_1(\mathrm{k})/\mathrm{Pe}^{2/3}, \tag{21}$$

which, at $\mathrm{k} = 0$, agrees satisfactorily with the expression obtained by Natanson for convective diffusion to a cylinder ([5]).

Thus, taking into account the joint action of inertial effects and diffusion leads to the manifestation of a number of new effects and can substantially change the magnitude of the capture coefficient. This circumstance is of considerable importance, since it may contribute to a change in our ideas about certain processes of capture of aerosol particles.

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* The value of n_1 was calculated from equation (1) on the BESM-2 electronic computer.

Note: Figure translations are in progress. See original paper for figures.