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Abstract

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MATHEMATICS

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ON NONNORMAL REGULAR PARTITIONS OF EUCLIDEAN SPACE

(Presented by Academician A. D. Aleksandrov, 5 X 1964)

1. A partition of n-dimensional Euclidean space into finite convex polyhedra is called convex. We shall call it **regular** if an arbitrarily chosen polyhedron of the partition can be mapped onto any polyhedron of this partition by a motion of the space (of the first or second kind) that maps the partition onto itself (i.e., if the group of motions of the space mapping the partition onto itself is transitive with respect to its polyhedra). We shall call it **normal** if every $(n-1)$-dimensional face of any polyhedron of the partition is a face of some other polyhedron of this partition (i.e., the polyhedra are adjacent along whole faces).

In the definition of a normal partition it is essentially understood that the different $(n-1)$-dimensional faces of each polyhedron lie in different $(n-1)$-dimensional planes (i.e., dihedral angles are not flattened). Without this condition, any convex partition could be turned into a normal one by calling each intersection of exactly two distinct polyhedra of the partition an $(n-1)$-dimensional face.

In recent years normal regular partitions of Euclidean spaces have been studied in sufficient detail (^{1}). For $n = 2$, the theory of regular partitions itself has been constructed purely topologically, in connection with which the requirement of normality of the partition became superfluous; all topologically distinct types of regular partitions of the plane have been found (^{2}). For $n > 2$, a substantive theory has been constructed only for normal regular partitions. Namely, B. N. Delone and N. N. Sandakova developed a general theory of Dirichlet partitions (^{1,3}) and gave a finite algorithm for deriving all topologically distinct types of these partitions for any $n \geqslant 3$. For the more general case of partitions, B. N. Delone proved the so-called fundamental theorem of the theory of stereohedra: the number of topologically distinct types of regular normal partitions of n-dimensional Euclidean space is bounded in terms of n alone (^{1,4}). The requirement of normality of the partition is used essentially in the proof of Delone' s theorem, and even for $n = 3$ it has not so far been possible to get rid of this requirement.

In the present note we give 4 infinite series of topologically distinct regular convex partitions of three-dimensional Euclidean space that do not satisfy the normality requirement. From these examples it follows that for $n \geqslant 3$ the requirement of normality in the fundamental theorem of the theory of stereohedra is of principal importance and cannot be replaced by a weaker topological requirement.

2. Take an arbitrary "brick" —a rectangular parallelepiped in three-dimensional Euclidean space—and denote by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ an orthogonal triple of vectors represented by its edges; the moduli a, b, c of these vectors will be called, respectively, the length, width, and height of the "brick." We shall further require that the ratio of the length to the width $a : b$ be equal to $N : 1$, where N is an arbitrary natural number greater than 2.

Subjecting the chosen "brick" to parallel translations by all possible vectors that are multiples of the vector of length $\mathbf{a}$, we obtain a linear row (strip-

iv) of "bricks" adjoining one another by entire "ends." Subjecting the resulting strip to translations by vectors that are multiples of the vector $b_1 = (1/N)a + b$, we obtain a plane layer of "bricks," consisting of a series of strips adjoining one another by "sides," with the "bricks" of adjacent strips not adjacent along whole faces— "sides" —but shifted with respect to one another by $1/N$ of the length.

We shall carry out the next step in four ways: 1) rotate the obtained layer by 180° about the line with direction vector b_1 passing through the "upper" vertex of one of the "bricks" (it passes along the plane—the "roof" of the layer—through the vertices of an infinite row of adjacent "bricks"); 2) proceed analogously to method 1), but choose the line—the axis of rotation—to pass through the middle of the "upper end" edge (representing the vector b); 3) subject the layer to a screw motion about the same line as in case 1), with a rotation through 180° and screw translation by the vector $\frac{1}{2}b_1$; 4) proceed analogously to method 3), choosing as the axis of the screw motion the line of case 2). In all four cases we obtain a second layer of "bricks" overlying the first layer along its whole "roof," while the lengths of the "bricks" of the second layer are perpendicular to the lengths of the "bricks" of the first layer. Each "brick" of the first layer borders on $N+4$ "bricks" of the second layer in cases 1) and 4), and on $N+3$ and $N+2$ "bricks" of the second layer in cases 2) and 3), respectively; moreover, the topologies of the adjacency of the layers are different in all four cases.

Finally, subject the obtained pair of adjacent layers to parallel translations by vectors that are multiples of the doubled height vector $2c$. We obtain a partition of space into equal "bricks" ; from the method of its construction it is clear that it is regular.

Each "brick" has 6 neighbors in its own layer, $N+4$, $N+3$, or $N+2$ neighbors in the layer adjoining it from above, and the same number in the layer adjoining it from below. The total number of neighbors corresponding to one "brick" is $2N+14$ in cases 1) and 4), $2N+12$ in case 2), and $2N+10$ in case 3). Since N may be varied arbitrarily, each of the four cases gives an infinite series of topologically distinct partitions.

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It follows from this as well that without the requirement of normality the theorem of B. N. Delone (^{4}) would not be true.

It is quite clear how to construct the simplest multidimensional generalizations of these partitions. It suffices to take n-dimensional "bricks," impose the same requirement on the first two dimensions ($a : b = N : 1$), and construct first a linear row, then a plane layer and a pair of layers; the obtained pair of layers, infinite in two dimensions, is then multiplied by parallel translations by vectors that are multiples of the edge-vectors in the remaining $n-2$ dimensions.

3. Let us consider the Fedorov groups that are full groups of motions carrying the obtained partitions of three-dimensional space onto themselves. For all four constructed series of partitions these groups belong to the orthogonal syngony (^{5}, ^{6}). The groups of the partitions constructed by methods 1) and 3) belong to the crystallographic class C_{2v}, and the groups of partitions 2) and 4) to the class D_{2h}. In all cases, to each even value of N there corresponds an empty Bravais parallelepiped (primitive lattice), and to each odd value a base-centered one. For each of the four series of partitions we obtain, respectively for even and odd N, the Fedorov groups: 1) $Pma2$ and $Ama2$ (according to Fedorov $6h$ and $12h$); 2) $Pccm$ and $Cccm$ ($17h$ and $20h$, according to Fedorov); 3) $Pmn2_1$ ($10a$ according to Fedorov) and $Ama2$; 4) $Pmna$ ($15a$ according to Fedorov) and $Cccm$.

Since under affine transformations of space preserving the symmetry elements the Fedorov group of the partition does not change (^{6}), it is not difficult to note that the initial "bricks" for constructing the partitions 1)—4) need not be chosen rectangular—it suffices to choose them straight (the vectors a and b may fail to be orthogonal, but the equality $a : b = N : 1$ must be observed). Such a picture will be obtained if one sub-

return our partitions by a contraction of the space to the plane with directing vectors $\mathbf{b}_1$ and $\mathbf{c}$.

Let us also note that one can construct other series of partitions, topologically distinct from the four found, by choosing, for the construction of the second layer adjacent to the first, as the axis of rotation through 180° or as the screw axis an arbitrary line with directing vector $\mathbf{b}_1$, lying on the flat "roof" of the first layer. In this case, as a rule, one obtains the same Fedorov groups that correspond to partitions 1) and 3), and, as an exception, those that correspond to partitions 2) and 4).

4. In conclusion we note that the partitions constructed by methods 2) and 3) have the following property: two polyhedra having common points have a common plane region; three polyhedra having common points have a common segment; four distinct polyhedra have at most one common point. If the nonempty intersection of each two distinct polyhedra of such a partition is called a face, of three—an edge, and of four—a vertex, we obtain a partition into $(2N+12)$-hedra or into $(2N+10)$-hedra (with

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> degenerate edges and vertices), adjacent along whole faces. The partitions obtained will be simplicial (primitive in Voronoi’ s sense).

This remark once again confirms the essential nature of a purely geometric understanding of the normality requirement in the proof of the fundamental theorem of the theory of stereohedra (4).

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