CYBERNETICS AND CONTROL THEORY

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Abstract

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ON THE COMPLEXITY OF GATE CIRCUITS REALIZING BOOLEAN MATRICES WITH UNDETERMINED ELEMENTS

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In applications one often encounters the problem of synthesizing control systems with an incomplete set of input values, i.e., systems in which certain combinations of input values do not occur (“forbidden” ones). The functioning of systems for forbidden input combinations is not specified and may be arbitrary. This creates additional possibilities for minimizing systems. Below, gate circuits are considered from this point of view; for them a synthesis method is obtained, and it is shown that the complexity of circuits is determined asymptotically only by the number of cells in which the conductivity matrix is specified, and does not depend on their arrangement.

1°. We extend the class of Boolean sets (vectors, matrices) by adjoining to the list of Boolean elements $\{0, 1\}$ an undetermined element $1/2$. By a **completion** of a set A we shall mean any purely Boolean set obtained from A by replacing each undetermined element by an arbitrary Boolean element. We identify n-element purely Boolean sets with the vertices of the n-dimensional unit cube, and n-element Boolean sets containing m undetermined elements $(0 \leq m \leq n)$ with the m-dimensional faces of this cube. We shall say that a vertex **pierces** a face containing it.

A set $\mathfrak{M}$ of vertices of the n-dimensional cube will be called **piercing** with respect to the set of all m-dimensional faces of this cube if every m-dimensional face is pierced by at least one vertex contained in $\mathfrak{M}$; the minimum number of vertices forming a piercing set will be denoted by $K(n, m)$.

Lemma.

$$2^{n-m} \leq K(n, m) \leq (n + 1)2^{n-m}.$$

Proof. In all there are $C_n^m 2^{n-m}$ faces of dimension m in the n-dimensional cube, and each vertex pierces C_n^m faces. This implies the lower bound.

Upper bound. Form the table $\|a_{i,j}\|$, where $i = 1, \dots, 2^n$ and $j = 1, \dots, C_n^m 2^{n-m}$; $a_{i,j} = 1$ if the i-th vertex pierces the j-th face, and $a_{i,j} = 0$ otherwise.

Choose k vertices piercing the maximum number of faces; denote by γ_k the fraction of pierced faces. The unpierced faces give $(1-\gamma_k)C_n^m 2^n$ ones in the table $\|a_{i,j}\|$, and therefore there is a row in it containing at least $\rfloor(1-\gamma_k)C_n^m\lceil^*$ of these ones. This means that one can choose a $(k+1)$-st vertex piercing $\rfloor(1-\gamma_k)C_n^m\lceil$ new faces. Therefore

$$\gamma_{k+1} \geq \frac{1}{2^{n-m}} + \left(1 - \frac{1}{2^{n-m}}\right)\gamma_k.$$

From the last inequality, by induction on k, we obtain (taking into account that $\gamma_1 = 1/2^{n-m}$)

$$\gamma_k \geq 1 - \left(1 - \frac{1}{2^{n-m}}\right)^k.$$

$*$ $\rfloor a\lceil$ denotes the least integer not smaller than the number a.

Denote by N the number

$$\left[\frac{\lg_2 C_n^m}{\lg_2 \dfrac{1}{1-1/2^{n-m}}}\right].$$

For $k = N$ no more than 2^{n-m} faces remain uncovered. Therefore $K(n,m) \leq N + 2^{n-m} \leq (n+1)2^{n-m}$.* The lemma is proved.

2°. We shall say that a gate circuit **realizes** a matrix A if it realizes, in the usual sense, some completion of the matrix A. The matrix obtained from A by replacing all Boolean elements by zeros and all undefined elements by ones will be called the **modulus** of the matrix A. Denote by $\mathfrak{B}(p,q,R)$ the class of all (p,q)-matrices** having modulus R. As usual (^{1}), associate with the class of matrices $\mathfrak{B}$ the Shannon function $B_m(\mathfrak{B})$—the minimum number of gates sufficient for realizing every matrix from $\mathfrak{B}$ by a gate circuit of depth not exceeding m.

Theorem. *Let a sequence of classes $\mathfrak{B}(p_n, q_n, R_n)$ be such that the fraction of ones in R_n is β_n and the following conditions are satisfied:*

a) $q_n \leq p_n$;

b) $q_n \to \infty$;

c) $(1-\beta_n)q_n / \lg_2 p_n \to \infty$;

d) $\lg_2 \dfrac{1}{1-\beta_n} / \lg_2 p_n \to 0$.

Then

$$B_2(\mathfrak{B}(p_n, q_n, R_n)) \sim (1-\beta_n)\frac{p_n q n}{\ln_2 p_n}.$$

Proof. The lower bound is the cardinality bound (^{1}) and is determined by the fact that the class $\mathfrak{B}(p, q, R)$ consists of $2^{(1-\beta)pq}$ matrices***.

Upper bound. Let $A \in \mathfrak{B}(p, q, R)$. Introduce a positive integer parameter ξ and split the matrix A by columns into $]q/\xi[$ nonoverlapping strips, each of which has p rows and no more than ξ columns. Introduce a positive integer parameter μ and split every row of each strip into nonoverlapping elementary vectors, each of which contains exactly μ Boolean elements, except, possibly, for one vector containing a smaller number of Boolean elements. The number of undefined elements in the vectors may be arbitrary. The total number of vectors obtained does not exceed

$$N = \frac{(1-\beta)pq}{\mu} + p\left]\frac{q}{\xi}\right[.$$

The set of elementary vectors of each strip is split into no more than $\mu + 1$ subsets, placing into one subset all vectors containing the same number of Boolean elements. Each subset is split into groups, placing into one group all vectors identically situated in their rows. Since the position of a vector in a row is determined by specifying its ends, one subset gives no more than ξ^2 groups, and the total number of groups does not exceed

$$P = (\mu+1)\xi^2\left]\frac{q}{\xi}\right[.$$

Complete every elementary vector in such a way that no more than

$$Q = (\xi+1)2^{\mu}$$

different completions are obtained for each group (lemma).

$*$ $\quad C_n^m \le 2^n, \quad \lg_2\frac{1}{1-1/2^{n-m}} > \lg_2\left(1+\frac{1}{2^{n-m}}\right) \ge \frac{1}{2^{n-m}}.$

$**$ By a (p, q)-matrix we mean any matrix having p rows and q columns.

$***$ The index n is omitted.

To each elementary vector we assign a gate issuing from the input pole whose number is the number of the row in which this vector is located. The outputs of all gates assigned to vectors of one group having the same completions are joined into a single node (altogether no more than PQ nodes). Each node is connected

Fig. 1

Figure 1: Fig. 1

by a bundle of no more than ξ gates to the output poles whose column numbers are those in which the units of the corresponding completion are located. The resulting circuit of depth 2 realizes the matrix A and contains no more than

$$S = N + PQ\xi.$$

Let, for example,

Fig. 1

$$A = \left\| \begin{array}{ccc} (0,\ 1,\ 1/2), & (1,\ 0), & (1/2) \\ (1/2,\ 1,\ 1), & (1,\ 1/2,\ 0) & \\ (1,\ 1), & (1,\ 1/2,\ 1), & (1) \\ (1,\ 1/2,\ 1), & (1,\ 0), & (0) \\ (1,\ 1), & (1,\ 1,\ 1/2), & (1) \\ (1/2,\ 1,\ 0), & (1,\ 0,\ 1/2) & \end{array} \right\|.$$

Here $\xi = 6$, i.e. the matrix consists of one strip, $\mu = 2$, and the parentheses show the partition of the rows into vectors.

As a completion we take the matrix

$$\left\| \begin{array}{ccc} (0,\ 1,\ 0), & (1,\ 0), & (0) \\ (1,\ 1,\ 1), & (1,\ 0,\ 0) & \\ (1,\ 1), & (1,\ 1,\ 1), & (1) \\ (1,\ 1,\ 1), & (1,\ 0), & (0) \\ (1,\ 1), & (1,\ 1,\ 1), & (1) \\ (0,\ 1,\ 0), & (1,\ 0,\ 0) & \end{array} \right\|.$$

The corresponding circuit is shown in Fig. 1*.

Put

$$\xi = \left[\frac{\lg_2^2 p}{1-\beta} \right], \qquad \mu = \left[\lg_2 p - 9 \lg_2 \lg_2 p - 4 \lg_2 \frac{1}{1-\beta} \right].$$

Then

$$S = (1-\beta) \frac{pq}{\lg_2 p} \left(1 + O\left(\frac{\lg_2 \lg_2 p}{\lg_2 p} + \frac{\lg_2 p}{(1-\beta) q} + \frac{\lg_2 \frac{1}{1-\beta}}{\lg_2 p} \right) \right).$$

The theorem is proved.

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1. O. B. Lupanov, DAN, **111**, No. 6, 1171 (1956).

* After the partition of the rows into vectors and the completion of the matrix have been established, the circuit is synthesized uniquely, and its subsequent simplification lies outside the framework of the synthesis method. Therefore, in the circuit shown in Fig. 1, superfluous gates have been retained.

Note: Figure translations are in progress. See original paper for figures.

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