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Abstract

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FORMAL DYNAMIC MODEL OF UNITARY SYMMETRY

It is well known that the n^2-fold degeneracy of the terms of the hydrogen atom is connected with the fact that the Hamiltonian of the three-dimensional Coulomb problem $H_0 = p^2/2 - 1/r$ commutes not only with the three components of the angular-momentum tensor M_{ik}, but also with three additional operators

$$A_k = \frac{1}{2}(p_i M_{ik} + M_{ik} p_i) + \frac{q_k}{r} \quad (i, k = 1, 2, 3). \tag{1}$$

Forming the operators $M_{k4} = A_k(-2H_0)^{-1/2}$, we find that the 6 quantities M_{ik}, M_{i4} constitute an antisymmetric tensor of generators of the group O_4.

It is known that the representations of the group O_4 can be classified by two numbers k and l, the dimension of the representation being equal to $(2k+1)(2l+1)$. Therefore, in the three-dimensional Coulomb problem not all representations of O_4 are realized, but only the "diagonal" ones, for which $2k+1 = 2l+1 = n$. Let us also note that the number of harmonic polynomials of degree n on a sphere in a space of 4 dimensions is precisely equal to n^2. The realization of the representations of the 3-dimensional Coulomb problem in the form of 4-dimensional spherical functions was obtained by V. A. Fock ([1]). It is important to note that the presence of only part of the representations (a "cut-out" of representations) is characteristic of problems with accidental degeneracy. The degeneracy of the equations of the isotropic three-dimensional oscillator corresponds to the representations $D(p, 0)$ of the group $SU(3)$.

The purpose of this note is to construct a quantum-mechanical model whose level degeneracy will correspond one-to-one to all representations of the group $SU(3)$, each of which will occur only once. Obviously, the model must possess no fewer than 5 degrees of freedom in terms of the number of quantum numbers that carry out the classification of states in the group $SU(3)$ (p, q, Q, Y, T).

We shall now make use of the circumstance ([2]) that the presence of accidental degeneracy in the problem with potential $V = 1/r = (q_1^2 + \ldots + q_N^2)^{-1/2}$ does not depend on the number of degrees of freedom N, the symmetry of the levels being determined by the group O_{N+1}. We shall therefore consider the "five-dimensional Coulomb problem"

$$\Omega_0\psi = \omega_0\psi; \qquad \Omega_0 = \frac{1}{2}\Delta_5 - \frac{1}{r}; \qquad r = (q_1^2 + \ldots + q_5^2)^{1/2}. \tag{2}$$

The symmetry group O_6 contains $SU(3)$ as a subgroup. Formula (1) as before defines the components of a conserved 5-vector, which, being normalized according to

$$M_{i6} = \frac{A_i}{\sqrt{-2\Omega_0}}, \quad i = 1, 2, \ldots, 5,$$

complete the antisymmetric angular-momentum tensor of the group O_5 to the angular-momentum tensor of the group O_6.

Calculations, entirely analogous to the calculations for the hydrogen atom, lead to the formula for the terms of problem (2)

$$\omega_0 = -1/2(n+2)^2,$$

where n is the principal quantum number, taking nonnegative integer values. The degree of degeneracy of these terms will be

$$M(n) = (n+1)(n+2)(n+3)(2n+4)/4!.$$

Let us now note that the multipletness $M(n)$ is expressed in terms of the multipletness $M(p,q)$ of the representations $D(p,q)$ of the group $SU(3)$ by the formula

$$M(n) = \sum_{\substack{p,q \\ p+q=n}} M(p,q). \tag{3}$$

This formula shows that each $M(n)$-fold degenerate term of problem (2) consists of a set of $M(p,q)$-fold degenerate $SU(3)$ terms such that $p+q=n$. The ground state $n=0$ contains the singlet $D(0,0)$. The first excited state $n=1$ consists of two triplets $D(1,0)$ and $D(0,1)$; the state $n=2$ consists of the octet $D(1,1)$ and two sextets $D(0,2)$ and $D(2,0)$, etc.

Just as in the hydrogen-like atom the deviation from Coulomb' s law removes the l-degeneracy, so in our case one can break the symmetry O_6 by introducing additions into the Hamiltonian (2). Thus, for example, by adding an arbitrary function of the radius vector r, we remove the accidental degeneracy, lowering the symmetry to the group O_5. Our task is to lower the symmetry to the group $SU(3)$, i.e., to split the $M(n)$-fold degenerate terms into $n+1$ multiplets with multiplicity $M(p,q)$ $(p+q=n)$.

The generators of the group $SU(3)$, satisfying the commutation relations

$$[A_{\alpha\beta}, A_{\gamma\delta}] = \delta_{\alpha\delta} A_{\gamma\beta} - \delta_{\gamma\beta} A_{\alpha\delta},$$

can be constructed from the operators M_{ik} of the group O_6 in the following way:

$$A_{\alpha\beta} = -\left(n_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta} k\right) - i m_{\alpha\beta} \qquad (\alpha, \beta = 1, 2, 3), \tag{4}$$

where

$$n_{\alpha\beta} = \frac{1}{2}(M_{\alpha,\beta+3} + M_{\beta,\alpha+3}),$$

$$m_{\alpha\beta} = \frac{1}{2}(M_{\alpha\beta} + M_{\alpha+3,\beta+3}),$$

$$k = \mathrm{Sp}\, n = M_{14} + M_{25} + M_{36}.$$

The operator k commutes with all the operators $A_{\alpha\beta}$ and therefore is a number, expressed through the usual indices of the representations of $SU(3)$ by the formula

$$k = p - q. \tag{5}$$

Since the initial Hamiltonian Ω_0 commutes with all $A_{\alpha\beta}$, it is also a number on the group $SU(3)$. Its explicit expression in terms of representation indices,

$$\omega_0 = -1/2(p + q + 2)^2$$

can be obtained by considering the Casimir operator $C_2 = A_{\alpha\beta} A_{\beta\alpha}$ taking formula (5) into account.

It is now clear that if we add to the Hamiltonian an arbitrary function of the operators k and Ω_0,

$$\Omega_0 \to \Omega = \Omega_0 + f(k, \Omega_0),$$

then thereby we lower the symmetry to the group $SU(3)$ and obtain new terms

$$\Omega \psi_{pq} = \omega_{pq} \psi_{pq}, \qquad \omega_{pq} = -1/2(p + q + 2)^2 + f(p - q, \omega_0). \tag{6}$$

Of course, appropriate restrictions must be imposed on the function f so that, in particular, the spectrum (6) remains bounded from below. Additional restrictions on f can be obtained by considering the continuous spectrum.

Thus, we have obtained a spectrum whose terms are in one-to-one correspondence with the representations of the group $SU(3)$, i.e., to each representation $D(p, q)$ of the group $SU(3)$ there corresponds a term ω_{pq}, whose degeneracy is equal to

$$M(p, q) = (p + 1)(q + 1)(p + q + 2)/2.$$

Let us now discuss the possible meaning of the results obtained. We note that we did not invest any definite physical meaning in the initial dynamical variables q_i, p_i, or in the operator Ω. In fact, we used only the condition that the eigenfunctions ψ vanish as $q_i \to \infty$ in order to obtain the spectrum (i.e., in quantization). Therefore, if one attempts to describe elementary particles by means of models of this type, it is necessary to put concrete physical content into q_i, p_i, Ω, which will in turn lead to a relation between the particle masses and the eigenvalues ω_{pq}.

Thus, the construction presented should be regarded as a certain formal model of unitary symmetry.

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Note: Figure translations are in progress. See original paper for figures.