MATHEMATICS

1965

SovietRxiv

Abstract

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MATHEMATICS

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BEST QUADRATURE FORMULAS FOR CERTAIN CLASSES OF FUNCTIONS

(Presented by Academician I. N. Vekua on 16 XI 1964)

Let $W_{L_p}^{(r)}(M_r;0,1)$ $(p \geqslant 1)$ be the class of functions $f(x)$ having absolutely continuous derivatives of order $r-1$ and a derivative of order r satisfying the condition

$$\left\|f^{(r)}(x)\right\|_{L_p} = \left(\int_0^1 \left|f^{(r)}(x)\right|^p dx\right)^{1/p} \leqslant M_r \quad (p \geqslant 1). \tag{1}$$

In the case $p=\infty$, instead of condition (1) we assume that for the piecewise continuous derivative $f^{(r)}(x)$ the inequality

$$\left|f^{(r)}(x)\right| \leqslant M_r \quad (0 \leqslant x \leqslant 1) \tag{2}$$

holds.

The classes $W_{L_p}^{(r)}(M_r;0,1)$ in the cases $p=1,\infty$ are denoted, respectively, by $W_L^{(r)}(M_r;0,1)$ and $W^{(r)}(M_r;0,1)$.

S. M. Nikol' skii (^{1}), for functions $f(x)$ belonging to the class $W^r(M_r;0,1)$ and satisfying additionally the condition $f(0)=f'(0)=\cdots=f^{(r-1)}(0)=0$, constructed the best quadrature formula of the form

$$\int_0^1 f(x)\,dx \approx \frac{1}{r!}\sum_{k=0}^{N-1}\sum_{l=0}^{r-2} A_k^{(l)}(r-l-1)!\,f^{(l)}(x_k), \tag{3}$$

where $0 \leqslant x_0 \leqslant x_1 < \cdots < x_{N-1} \leqslant 1$.

In the present article we construct best quadrature formulas of the form (3), exact for polynomials of degree $r-1$, in the classes of functions $W^{(r)}(M_r;0,1)$, $W_L^{(r)}(M_r;0,1)$, and $W_{L_2}^{(r)}(M_r;0,1)$, where r is an even positive integer.

From the fact that the quadrature formula (3) is exact for polynomials of degree $r-1$, it follows that for a function $f(x) \in W^{(r)}(M_r;0,1)$ the inequality

$$|R(f)| = \left| \int_0^1 f(x)\,dx - \frac{1}{r!} \sum_{k=0}^{N-1} \sum_{l=0}^{r-2} A_k^{(l)} (r-l-1)!\, f^{(l)}(x_k) \right| \leqslant \frac{M_r}{r!} \int_0^1 |K(t)|\,dt,$$

where

$$K(t) = (1-t)^r - \sum_{k=0}^{N-1} \sum_{l=0}^{r-2} A_k^{(l)} E_{r-l}(x_k - t);$$

$$E_r(u) = \begin{cases} u^{r-1}, & \text{for } u > 0, \\ 0, & \text{for } u \leqslant 0. \end{cases}$$

Hence it follows that

$$\mathcal{E}_N\big(W^{(r)}\big) = \min_{(A_k^{(l)},\, x_k)} \left\{ \sup_{f \in W^{(r)}} |R(f)| \right\} = \frac{M_r}{r!} \min_{(A_k^{(l)},\, x_k)} \int_0^1 |K(t)|\,dt, \tag{4}$$

where the function $K(t)$ on the intervals $[0, x_0]$ and $[x_{N-1}, 1]$ is defined by the equality

$$K(t) = \begin{cases} t^r, & \text{for } 0 \leqslant t \leqslant x_0, \\ (1-t)^r, & \text{for } x_{N-1} \leqslant t \leqslant 1. \end{cases}$$

Let $x_k - x_{k-1} = 2h_k$ and $c_k = (x_{k-1} + x_k)/2$ $(k = 1, 2, \ldots, N-1)$. Note that the polynomial $g_k(x)$ with coefficient of x^r equal to one, and least deviating from zero in the metric of the space L on the interval (x_{k-1}, x_k), has the form

$$g_k(x) = h_k^r Q_r\left(\frac{x - c_k}{h_k}\right),$$

where

$$Q_r(x) = \frac{\sin[(r+1)\arccos x]}{2^r\sqrt{1-x^2}} \quad (-1 \leqslant x \leqslant 1).$$

In order that the polynomials $g_k(x)$ and $g_{k+1}(x)$ coincide at the point $x_k = c_k + h_k = c_{k+1} - h_{k+1}$, it is necessary and sufficient that the condition $h_k = h_{k+1}$ be satisfied $(k = 1, 2, \ldots, N-2)$. Further, it is not difficult to show that

$$h = 2x_0/\sqrt[r]{r+1}; \qquad x_0 = 1 - x_{N-1}; \tag{5}$$

whence, and from the equality $x_{N-1} = x_0 + (N-1)2h$, we find

$$x_k = (\sqrt[r]{r+1} + 4k)\omega_N \qquad (k = 0, 1, \dots, N-1), \tag{6}$$

where

$$\omega_N = [2\sqrt[r]{r+1} + 4(N-1)]^{-1}, \qquad h = 2\omega_N. \tag{7}$$

The coefficients $A_k^{(l)}$ are determined analogously to how this was done by S. M. Nikol' skii $(^1)$:

$$A_k^{(2i+1)} = 0 \qquad (i = 0, 1, \dots, (r-4)/2),$$

$$A_k^{(2i)} = \frac{2h^{2i+1}}{(r-2i-1)!} Q_r^{(r-2i-1)}(1) \qquad (i = 0, 1, \dots, (r-2)/2) \tag{8}$$

$$(k = 1, 2, \dots, N-2),$$

$$A_0^{(l)} = A_{N-1}^{(l)} = \frac{h^{l+1}}{(r-l-1)!} \left(\frac{r!}{(l+1)!} [Q_r(1)]^{(l+1)/r} + (-1)^l Q^{(r-l-1)}(1) \right).$$

It can be proved that only for the coefficients $A_k^{(l)}$ and the nodes x_k determined respectively by equalities (8) and (6) does the integral $\int_0^1 |K(t)|\, dt$ attain its minimum. Thus, the following assertion holds:

1. *The quadrature formula of the form (3), exact for polynomials of degree* $(r-1)$, *whose coefficients* $A_k^{(l)}$ *and nodes* x_k *are determined respectively by equalities (6) and (8), is the unique best quadrature formula for functions from the class* $W^{(r)}(M_r; 0, 1)$. *Moreover,*

$$\mathcal{E}_N(W^{(r)}) = \frac{M_r}{r!}\, \omega_N^r.$$

By similar reasoning the following assertions are proved:

2. *The quadrature formula of the form (3), exact for polynomials of degree* $r-1$, *whose coefficients* $A_k^{(l)}$ *are determined by equalities (8), and the nodes* x_k $(0 \leqslant x_0 < x_1 < \dots < x_{N-1} \leqslant 1)$ *by the equalities*

$$x_k = (1 + 2k\sqrt[r]{2^{r-1}})\omega_N \qquad (k = 0, 1, \dots, N-1),$$

where

$$\omega_N = \frac{1}{2}\left[1 + \sqrt[r]{2^{r-1}(N-1)}\right]^{-1}, \qquad h = \sqrt[r]{2^{r-1}}\omega_N,$$

is the unique best quadrature formula for functions from the class $W_L^{(r)}(M_r; 0, 1)$ and, moreover,

$$\mathcal{E}_N\left(W_L^{(r)}\right) = \frac{M_r}{r!}\omega_N^r.$$

Let us note that in the proof of this assertion the polynomial $Q_r(x)$, used in the proof of the first assertion, is replaced by the Chebyshev polynomial of the first kind $T_r(x)$.

3. The quadrature formula of the form (3), exact for polynomials of degree $r-1$, and whose coefficients $A_k^{(l)}$ are determined by the equalities (8), while the nodes x_k are given by the equalities

$$x_k = \left(1 + k\sqrt[r]{\frac{(2r)!}{(r!)^2}}\right)\omega_N,$$

where

$$\omega_N = \left[2 + \sqrt[r]{\frac{(2r)!}{(r!)^2}}(N-1)\right]^{-1}, \qquad h = \frac{1}{2}\sqrt[r]{\frac{(2r)!}{(r!)^2}}\omega_N,$$

is the unique best quadrature formula for functions from the class $W_{L_2}^{(r)}(M_r; 0, 1)$, with

$$\mathcal{E}_N\left(W_{L_2}^{(r)}\right) = \frac{M_r}{r!}(2r+1)^{-1/2}\omega_N^r$$

and $Q_r(x)$ is replaced by a polynomial of the form

$$X_r(x) = \frac{(r!)^2}{(2r)!}\left[\sum_{k=0}^{r/2}(-1)^k\frac{(2r-2k)!}{(r-2k)!(r-k)!k!}x^{r-2k}\right].$$

Let us note that from assertions 1 and 2, for $r = 2$, follow the best quadrature formulas constructed by T. A. Shaidaeva (^{3}). Moreover, assertion 3 for $r = 2$ coincides with the corresponding assertion of T. A. Shaidaeva in the case $p = 2$. Finally, assertion 3 in the case $r = 2$ is a direct generalization of the corresponding assertion of G. Ya. Doronin (^{2}).

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Received
11.XI.1964

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Note: Figure translations are in progress. See original paper for figures.

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