MATHEMATICS

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1965

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Abstract

Full Text

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A SINGULAR OPERATOR

IN A SCALE OF BANACH SPACES

1. Let S be a family of Banach spaces B_ρ depending on a real parameter $\rho > 0$. We denote the norm of an element $u \in B_\rho$ by $\|u\|_\rho$. The family S will be called a **scale of Banach spaces***, if the following condition is satisfied: for any ρ and $\rho' < \rho$,

$$B_{\rho'} \supset B_\rho, \qquad \|u\|_{\rho'} \leq \|u\|_\rho \quad (u \in B_\rho). \tag{1}$$

We shall call a linear operator L an **operator of type** D if it "shifts continuously" along the scale S, i.e., if for any ρ and $\rho' < \rho$ the operator L acts as a bounded operator from B_ρ into $B_{\rho'}$. Denote by $N(\rho, \rho')$ the norm of L as an operator from B_ρ to $B_{\rho'}$, so that for all $u \in B_\rho$,

$$\|Lu\|_{\rho'} \leq N(\rho, \rho')\|u\|_\rho \qquad (\rho' < \rho). \tag{2}$$

It is easy to see that the function $N(\rho, \rho')$ is nondecreasing both as ρ' increases and as ρ decreases (if the value of the other argument is fixed). It is natural to call the operator L **bounded in the scale** S if the norm $N(\rho, \rho')$ is bounded by a number independent of ρ, ρ'.

Definition. An operator L of type D is called **singular in the scale** S if it is not bounded in the scale S, but there exists a constant ω such that for any ρ and $\rho' < \rho$,

$$(\rho - \rho')N(\rho, \rho') < \omega. \tag{3}$$

In what follows we shall consider operators L_t depending on a parameter $t \geq 0$, which for every t are operators of type D. The operator L_t will be called **continuous in** t if, for any fixed $u \in B_\rho$, the element $L_{tu} \in B_{\rho'}$ depends continuously on t in the norm $\|u\|_{\rho'}$, and if the estimate of the norm of the operator L_t does not depend on t.

Consider the following Cauchy problem: find a function $u = u(t) \in B_\rho$ such that

$$\frac{du}{dt} = L_{tu}, \qquad u(0) = u_0 \in B_{\rho_0}. \tag{4}$$

Theorem 1. *If L_t is a singular operator in the scale S and is continuous in t, then for every $\rho < \rho_0$ problem (4) has a unique solution $u(t) \in B_\rho$, defined on the interval*

$$0 \le t < \frac{\rho_0}{e\omega}\left(1 - \frac{\rho}{\rho_0}\right), \tag{5}$$

where ω is the constant from estimate (3).

This theorem is proved by the method of successive approximations. The nontrivial point is the estimate of convergence, which turns out to be power-like. If, however, problem (4) is considered with an operator L_t bounded in the scale S, then it will have a solution for all $t \ge 0$ (thanks to the exponential estimate of convergence).

* A narrower concept of a **normal scale** of Banach spaces was considered in (^{1}).

2. In applications to differential equations, scales S are important in which the singular operator is the differentiation operator. An example of such a scale is the scale of analytically continuable functions. The construction of the scale of analytically continuable functions is based on some Banach space A_0 of m-component vector-functions $u(x)$ (in what follows, for brevity, we shall simply speak of functions $u(x)$), defined on some domain G of the Euclidean space $E^n(x)$ of points $x = (x^1, \dots, x^n)$. It is assumed that A_0 contains all functions $u(x)$ which are analytically continued to functions $u(z)$ $(z = x + iy)$, analytic in some domain of the space $E^{2n}(x, y)$ containing the closure $\overline{G}$. Further, let $D_i = \partial/\partial x^i$ $(i = 1, \dots, n)$. The metric of A_0 is also required to satisfy the following condition: if a sequence $\{u_r\}_1^\infty$, $u_r \in A_0$, is such that $D_i u_r \in A_0$, then

$$\text{from} \quad u_r \xrightarrow[A_0]{} u, \qquad D_i u_r \xrightarrow[A_0]{} v_i \quad \text{it follows that} \quad v_i = D_i u. \tag{6}$$

A function $u(x) \in A_0$ will be called **analytically continuable** if it has derivatives of all orders and if there exists $\rho > 0$ such that

$$\|D^\alpha u\|_0 < K\alpha!\,\rho^{-|\alpha|} \tag{7}$$

with a constant K independent of α. In (7) the usual notation is used: the integer vector $\alpha = (\alpha_1, \dots, \alpha_n)$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$. The symbol $\|\cdot\|_0$ denotes the norm in A_0.

The linear system of functions $u(x) \in A_0$ for which (7) is fulfilled for a fixed ρ will be denoted by A_ρ. On the set A_ρ we introduce the norm*

$$\|u\|_\rho = \sup_\alpha \frac{\rho^{|\alpha|}}{\alpha!} \|D^\alpha u\|_0, \tag{8}$$

which turns A_ρ into a linear normed space.

Lemma 1. *The space A_ρ with norm (8) is complete.*

This property follows from the completeness of A_0 and condition (6). The family of spaces A_ρ satisfies condition (1) and, consequently, forms a scale of Banach spaces. This scale, which we shall denote by $S(A_0)$, is the scale of analytically continuable functions.

Analogously one can construct scales of analytically continuable functions on manifolds $\mathfrak{M} \subset E^n$.

Lemma 2. *The differentiation operators D_i are singular in the scale $S(A_0)$.*

The assertion of Lemma 2 follows at once from the fact that for any $\rho' < \rho$

$$\sup_{k=1,2,\dots} k \left(\frac{\rho'}{\rho}\right)^k < \frac{1}{e \log \rho/\rho'} = \frac{1}{e} \beta \left(\frac{\rho'}{\rho}\right) \frac{\rho'}{\rho - \rho'},$$

where $\beta(\xi) \to 1$ as $\xi \to 1 - 0$.

Let us note that condition (6) is fulfilled for the usual spaces A_0: the space $C(\overline{G})$ of continuous functions or the space $L^p(G)$ of functions integrable to the power $p > 1$. This gives concrete scales $S(C)$ and $S(L^p)$. The spaces A_ρ in these scales will be denoted respectively by C_ρ and L^p_ρ. It can be shown that the strict embedding $C_\rho \subset L^p_\rho$ holds.

As an example of an application of Theorem 1, consider the following Cauchy problem:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^n a^i D_i u + a^0 u, \qquad u|_{t=0} = u_0(x). \tag{9}$$

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* This norm was also considered in (2).

for an m-dimensional vector function $u = u(x,t)$, where $a^j = a^j(x,t)$ $(j = 0, 1, \dots, n)$ are given square matrices of order m, and $u_0(x)$ is a given vector function, regarded as functions of the point $x \in G$ and of the parameter $t \geqslant 0$.

Theorem 2. If the row vectors of the matrices a^j and the vector u_0 belong to C_{ρ_0} and are continuous in t (as elements of C_{ρ_0}) for $t \geqslant 0$, then problem (9) has a unique solution $u(x,t) \in C_\rho$ for every $\rho < \rho_0$, defined on the interval

$$0 \leqslant t < \frac{q}{M} \left(1 - \frac{\rho}{\rho_0}\right)^2, \tag{10}$$

where the constant q depends only on ρ_0, while M is proportional to the norm of the matrices a^j in C_{ρ_0}.

This theorem is in fact a consequence of Theorem 1. The difference between the estimates (10) and (5) arises from the fact that in (9), in addition to the singular operators D_i, there are also operators of multiplication by the matrices a^j.

Remark 1. Applying Holmgren' s method, one can prove that the uniqueness of the solution of problem (9) under the assumptions of Theorem 2 is guaranteed in a broader class, namely in the class of functions $u(x, t)$ continuously differentiable once with respect to x, t.

Remark 2. If in problem (9) the coefficients a^j are holomorphic functions of t, then the method based on considering this problem in the scale $S(C)$ also makes it possible to prove an existence theorem for a solution holomorphic in t.

Of course, in the case of problem (9), Remark 2 gives no new result, since this is the classical Cauchy theorem on holomorphic solutions. In the next section another problem is considered, for which the analogue of the Cauchy theorem, apparently, was not previously known.

3. The following problem served as the source for the theory set forth above.

In the unit disk it is required to find a harmonic function $u = u(r, \theta, t)$ (r, θ are polar coordinates) of class C^1 ($r \leqslant 1,\ t \geqslant 0$), depending on the parameter t and satisfying the conditions

$$\partial u / \partial t = a u_\theta + b u_r + c u \quad (r = 1), \qquad u(1, \theta, 0) = u_0(\theta), \tag{11}$$

the second of which is the initial condition. Here a, b, c, u_0 are given functions of the variables θ, t.

This problem, in contrast to problem (9), is already essentially nonlocal. As simple examples show, it is ill-posed in the sense that its solution, generally speaking, does not exist. For example, if the first condition (11) has the form $u_t = u_x$, then the solution can only be the function $u = u_0(x + t, y)$. In order that this solution be a harmonic function in the disk $r < 1$, it is necessary that the initial harmonic function $u_0(x, y)$ admit an analytic continuation beyond the unit disk. However, in the class of analytically continuable functions a, b, c, u_0, problem (11) turns out to be posed correctly.

Let us introduce the scales $S(C)$, $S(L^2)$ of analytically continuable functions $u(\theta)$, defined and 2π-periodic on $E^1(\theta)$, with the norm in L^2 understood as the norm of $L^2(0, 2\pi)$. We denote the norm in the corresponding space C_ρ by $\| \cdot \|_\rho$.

Theorem 3. If $a, b, c, u_0 \in C_{\rho_0}$ and are continuous in t (as elements of C_{ρ_0}) for $t \geqslant 0$, then problem (11) has a unique solution $u(1, \theta, t) \in L^2_\rho$ for every $\rho < \rho_0$, defined on the interval

$$0 \leqslant t < \frac{q}{M}\left(1 - \frac{\rho}{\rho_0}\right)^2, \tag{12}$$

where the constant q depends only on ρ_0, and

$$M = \sup_t \max\{\|a\|_{\rho_0},\ \|b\|_{\rho_0},\ \|c\|_{\rho_0}\}.$$

The use of the scale $S(L^2)$ here is more convenient owing to the fact that for a harmonic function $u(r, \theta)$ the equality

$$\|u_\theta(1, \theta)\|_{L^2} = \|u_r(1, \theta)\|_{L^2}$$

holds.

Remarks 1 and 2 made in connection with Theorem 2 are also valid for Theorem 3. It is interesting that the estimate (10) in the local problem (9) and the estimate (12) in the nonlocal problem (11) are essentially identical.

In conclusion, let us note that among possible generalizations of Theorem 3, of interest is its extension to problems analogous to (11), but not for harmonic functions; rather, for solutions of an arbitrary elliptic equation with analytic coefficients.

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Received
8 V 1965

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Note: Figure translations are in progress. See original paper for figures.

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