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Abstract

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MATHEMATICS

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ON DOUBLE OPERATOR STIELTJES INTEGRALS

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1. In this paper we study operators in a Hilbert space $\mathfrak{H}$ that are given by integrals of the form

$$Q = \iint \varphi(\lambda, \mu)\, dF_\mu\, T\, dE_\lambda. \tag{1}$$

Here F_μ, E_λ are any two orthogonal resolutions of the identity; T is a bounded operator in $\mathfrak{H}$; $\varphi(\lambda, \mu)$ is a complex function. Below, as usual, $\mathfrak{R}$ denotes the ring of all bounded operators in $\mathfrak{H}$; $\mathfrak{S}_\infty$ is the ideal of completely continuous operators in $\mathfrak{H}$; $\mathfrak{S}_p$ $(1 \leq p < \infty)$ is the ideal of all operators from $\mathfrak{S}_\infty$ whose sequence of singular numbers belongs to l_p. In particular, $\mathfrak{S}_1$ is the class of nuclear operators; $\mathfrak{S}_2$ is the Hilbert-Schmidt class.

Integrals of the form (1) constitute a useful apparatus in a number of questions of operator theory. Apparently they first appeared in the work (^{1}) in connection with certain questions of the analytic theory of perturbations. In (^{1}) the simplest condition for convergence of such integrals in $\mathfrak{R}$ was indicated. In (^{2,3}) the significance of integrals (1) in the theory of perturbations on the continuous spectrum became clear. In particular, it became necessary to study these integrals in some of the classes $\mathfrak{S}_p$.

Our aim is to study systematically the properties of the operator Q as a function of the behavior of the function $\varphi(\lambda, \mu)$ and of the properties of the operator T. In doing so we clarify the question of the sense in which the integrals (1) are to be understood, and establish relations among the various possible definitions of the integral. We shall regard the integral (1) as an operator on T, acting from one of the classes $\mathfrak{S}_p$ $(1 \leq p \leq \infty)$, $\mathfrak{R}$ into another such class. (Following I. Ts. Gohberg and M. G. Krein (^{4}), operators in spaces of operators will be called transformers. If a transformer Φ is bounded as an operator from $\mathfrak{S}_p$ to $\mathfrak{S}_q$, then we shall call it a transformer of class $(\mathfrak{S}_p, \mathfrak{S}_q)$.)

We note that the study of the connection between the Hermitian components of a Volterra operator ($^{5-10}$) is related to the study of one special transformer of the form (1). Namely, the well-known formula

$$Q = 2i \int_0^1 E_\lambda T \, dE_\lambda,$$

which reconstructs the Volterra operator Q from its imaginary component T, is written by means of the integral (1) for $E_\lambda = F_\lambda$ and $\varphi(\lambda, \mu) = i[\operatorname{sign}(\lambda - \mu) + 1]$. It should be noted at once that the results of $(^{5-10})$ follow from our theorems only in the cases where $T \in \mathfrak{S}_2$ or $T \in \mathfrak{S}_1$. We also point out that some transformers studied in $(^4)$ can also be written in the form (1). There, however, somewhat different questions were considered. Finally, we note that multidimensional singular integrals with characteristic depending on the pole $(^{11})$ are essentially integrals of the form (1). In this case $T = I$, and both spectral families F_μ, E_λ depend on multidimensional points μ and λ. This question will be discussed in more detail elsewhere.

2. Let us first consider the integral (1) for $T \in \mathfrak{S}_2$. We shall use the fact that the class $\mathfrak{S}_2$ is a Hilbert space with respect to the scalar product $\langle T_1, T_2 \rangle = \operatorname{Sp} T_2^* T_1$. The families of transformations $\mathcal{F}_\mu, \mathcal{E}_\lambda$, defined by the formulas $\mathcal{F}_\mu T = F_\mu T$, $\mathcal{E}_\lambda T = T E_\lambda$, are decompositions of the identity in $\mathfrak{S}_2$. They obviously commute:

   $$\mathcal{F}_\mu \mathcal{E}_\lambda T = \mathcal{E}_\lambda \mathcal{F}_\mu T = F_\mu T E_\lambda.$$

   The commuting families $\mathcal{F}_\mu, \mathcal{E}_\lambda$ generate in the usual way $^{(12)}$ a spectral measure $\mathcal{G}(e)$ in the plane. Moreover, if $\varphi(\lambda, \mu)$ is $(\mathcal{G})$-measurable and bounded with respect to the measure $\mathcal{G}(e)$, i.e.

   $$\|\varphi\|_G \equiv (\mathcal{G}) - \sup |\varphi(\lambda, \mu)| < \infty, \tag{2}$$

   then the integral

   $$\Phi = \iint \varphi(\lambda, \mu) \, d\mathcal{G}(e) \tag{3}$$

   defines a normal operator (transformation) in $\mathfrak{S}_2$ with norm $\|\varphi\|_G$.

Definition. For any $T \in \mathfrak{S}_2$, the **double operator integral** (1) is called the value Q of the transformation (3) on the operator T: $Q = \Phi T$.

The validity of the following assertion is now evident.

Theorem 1. Let the $(\mathcal{G})$-measurable function $\varphi(\lambda, \mu)$ satisfy condition (2). Then for every $T \in \mathfrak{S}_2$ the operator integral (1) converges in the norm of $\mathfrak{S}_2$ and defines a transformation Φ of the class $(\mathfrak{S}_2, \mathfrak{S}_2)$. The set of transformations Φ of the form (3) forms a commutative normed ring, isomorphic and isometric to the ring of $(\mathcal{G})$-measurable functions with norm (2). Moreover, to the adjoint transformation Φ^* there corresponds the function $\overline{\varphi(\lambda, \mu)}$.

Let E_λ, F_μ correspond in $\mathfrak{H}$ to self-adjoint operators A, B. It is clear that the ring of transformations (3) coincides with the ring of bounded functions of the commuting (possibly unbounded) transformations of right multiplication by A and left multiplication by B in $\mathfrak{S}_2$.

3. In studying the integrals (1) in other classes we can use the definition of item 2 only for $T \in \mathfrak{S}_p,\ 1 \leq p \leq 2$. If the transformation $\Phi \in (\mathfrak{S}_1, \mathfrak{S}_1)$ corresponding to the function $\varphi(\lambda, \mu)$, then, obviously, the adjoint transformation $\Phi^* \in (\mathfrak{R}, \mathfrak{R})$. It is now simplest to define the integral (1) for $T \in \mathfrak{R}$ by the formula $Q = \Phi^* T$. It is clear that for $T \in \mathfrak{S}_\infty$ one also has $Q \in \mathfrak{S}_\infty$. In an analogous way one can define the integral (1) also in the case of the classes $\mathfrak{S}_p,\ p > 2$. Taking this definition as basic, we shall return to this question in item 4.

Theorem 2. Let the spectral family E_λ (F_μ) be constant outside a finite interval, and let the bounded Borel function $\varphi(\lambda, \mu)$ satisfy the condition $\operatorname{Lip}\alpha$ in the variable λ (μ), with a constant independent of μ (of λ). If $\alpha > 1/2$, then the transformation Φ defined by the integral (1) belongs to every class $(\mathfrak{S}_p, \mathfrak{S}_p)$, $1 \leq p \leq \infty$, and to the class $(\mathfrak{R}, \mathfrak{R})$. If $\alpha \leq 1/2$, then $\Phi \in (\mathfrak{S}_p, \mathfrak{S}_p)$, $2(1+2\alpha)^{-1} < p < 2(1-2\alpha)^{-1}$.

Remarks. 1) The case of an infinite interval of integration can be treated by means of a change of variable. 2) It can be shown that the results of Theorem 2 cannot be improved in terms of the Lipschitz classes. 3) For $\alpha = 1/2$ under the conditions of Theorem 2 one can additionally assert,* that $\Phi \in (\mathfrak{S}_1, \mathfrak{S}_\omega)$ and $\Phi \in (\mathfrak{S}_\omega, \mathfrak{S}_\infty)$. 4) The preceding assertion remains valid if the condition $\varphi \in \operatorname{Lip} 1/2$ is replaced by the condition of bounded variation of $\varphi(\lambda, \mu)$ with respect to the variable λ (μ), uniformly in μ (λ). In this case there is no need to assume the interval of variation $\lambda(\mu)$ is finite. 5) For $\alpha > 1/2$ the transformer Φ belongs to any class $(\mathfrak{S}, \mathfrak{S})$, where $\mathfrak{S}$ is an arbitrary minimal or maximal normed ideal in $\mathfrak{S}_\infty$. This fact follows directly from the analogue of B. S. Mityagin' s interpolation theorem $(^{16})$ for ideals in $\mathfrak{S}_\infty$.

4. Under the hypotheses of Theorem 2, the integral (1) can be given meaning as the limit of integral sums. Namely, if one introduces the operator-valued function

$$K(\lambda) = \int \varphi(\lambda, \mu)\, dF_\mu,$$

then the integral (1) can be written (for the time being formally) in the form of the repeated integral

[*] For the definition of the normed ideals $\mathfrak{S}_\omega, \mathfrak{S}_\Omega$, introduced by V. I. Matsaev, I. Ts. Gohberg, and M. G. Krein, see, for example, in $^{(7)}$.

$$Q = \int K(\lambda) T \, dE_\lambda. \tag{4}$$

The convergence of the integral (4) is ensured by the following theorem.

Theorem 3. Let the bounded Borel function $\varphi(\lambda, \mu)$ satisfy a $\mathrm{Lip}\,\alpha$ condition in the variable λ, with a constant independent of μ, and let the family E_λ be constant outside a finite interval. Then, for $T \in \mathfrak{R}$ when $\alpha > 1/2$, and for $T \in \mathfrak{S}_p$, $2 < p < 2(1-2\alpha)^{-1}$, when $\alpha \leq 1/2$, the integral (4), understood as the limit in $\mathfrak{R}$ of arbitrary integral sums of the form

$$\sum_{k=0}^{n-1} K(\tilde{\lambda}_k) T \, (E_{\lambda_{k+1}} - E_{\lambda_k}), \qquad \tilde{\lambda}_k \in [\lambda_k, \lambda_{k+1}), \tag{5}$$

exists and coincides with the integral (1), understood in the sense of the definition in § 3. An analogous assertion is valid upon interchanging the roles of the variables λ and μ.

If both families E_λ, F_μ are constant outside finite intervals and $\varphi(\lambda, \mu) \in \mathrm{Lip}\,\alpha$ jointly in the variables λ, μ, then the double integral sums

$$\sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \varphi(\lambda_{kl}, \mu_{kl}) \, (F_{\mu_{k+1}} - F_{\mu_k}) T (E_{\lambda_{l+1}} - E_{\lambda_l}) \tag{6}$$

converge in $\mathfrak{R}$ to the integral (1).

Let us also note that, under the hypotheses of Theorem 3, for $\alpha > 1/2$ and $T \in \mathfrak{S}_p$ $(1 \leq p \leq \infty)$, the integral sums of the form (5), (6) converge to the integral (1) in the norm of $\mathfrak{S}_p$.

The proof of Theorem 3 is based on certain results concerning the integration of additive set-functions not having bounded variation. In view of the lack of space, we give only the simplest assertion of this kind.

Theorem 4. Let $f(\Delta)$ be an additive vector-valued function of half-open intervals $\Delta \subset [a, b)$ with values in a Banach space $\mathfrak{X}$, such that for some $M, \gamma > 0$, for any partition of $[a, b)$ into non-overlapping intervals $\Delta_k = [\lambda_k, \lambda_{k+1})$, the inequality

$$\sum_{k=0}^{n-1} \| f(\Delta_k) \| (\lambda_{k+1} - \lambda_k)^\gamma \leq M$$

holds.

Furthermore, let $K(\lambda)$ be an operator-valued function in $\mathfrak{X}$ satisfying on $[a, b]$ the condition $\mathrm{Lip}\,\alpha$ $(\alpha > \gamma)$ in the operator norm. Then the integral

$$\int_a^b K(\lambda)f(d\lambda)$$

exists as a strong limit in $\mathfrak{X}$ of Riemann-Stieltjes integral sums.

Theorem 4 may be regarded as a generalization of V. T. Kondurar' s theorem (13) on the integrability of a function $f(t) \in \operatorname{Lip} \alpha$ with respect to a function $g(t) \in \operatorname{Lip} \beta$ when $\alpha + \beta > 1$.

5. For certain special classes of transformers the result of Theorem 2 can be sharpened.

Theorem 5. *Let the derivative* $\psi'(\lambda)$ *of the function* $\psi(\lambda)$ *satisfy a* $\operatorname{Lip} \alpha$ *condition for some* $\alpha > 0$*. Then the transformer*

$$\int_a^b \int_a^b [\psi(\lambda) - \psi(\mu)](\lambda - \mu)^{-1}\, dF_\mu T\, dE_\lambda$$

belongs to each of the classes $(\mathfrak{R}, \mathfrak{R})$, $(\mathfrak{S}_p, \mathfrak{S}_p)$, $1 \le p \le \infty$.

A consequence of Theorem 5 is

Theorem 6. *Let* U, V *be unitary operators in* $\mathfrak{H}$*, and let* $\xi(t)$ *be a function on the circle* $|t| = 1$ *whose derivative* $\xi'(t) \in \operatorname{Lip} \alpha$ $(\alpha > 0)$*. If* $V - U \in \mathfrak{S}_p$ $(1 \le p \le \infty)$*, then*

$$\xi(V) - \xi(U) \in \mathfrak{S}_p.$$

An analogous result is valid for functions of self-adjoint operators. The assertion of Theorem 6 for $p = 1$ is of definite interest for the abstract theory of scattering (2,3) and the theory of the spectral shift function (14,15). In particular, it is easy to show that, under the hypotheses of Theorem 6, the trace formula (14) remains valid for unitary operators.

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Note: Figure translations are in progress. See original paper for figures.

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