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Abstract

Full Text

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ON SOME CASES OF PRESERVATION OF CONDITIONALLY PERIODIC MOTIONS UNDER A SMALL CHANGE OF THE HAMILTONIAN FUNCTION

(Presented by Academician A. N. Kolmogorov, May 3, 1965)

In the present note there is contained the following theorem, generalizing A. N. Kolmogorov' s theorem ([1,2]) on the preservation of conditionally periodic motions under a small change of the Hamiltonian function.

Theorem. Consider the Hamiltonian system

$$\dot{x} = -\partial H/\partial y, \qquad \dot{y} = \partial H/\partial x \qquad (x = (x_1, \dots, x_m),\ y = (y_1, \dots, y_m)),$$

$$\dot{p} = -\partial H/\partial q, \qquad \dot{q} = \partial H/\partial p \qquad (p = (p_1, \dots, p_n),\ q = (q_1, \dots, q_n)) \quad (1)$$

with Hamiltonian function of the form

$$H = H_0(x, y, p) + H_1(x, y, p, q),$$

possessing the following properties:

1. The function $H_0(x, y, p)$ depends analytically on x, y, p in the domain

$$|x| < h, \quad |y| < h, \quad p \in G \quad (h > 0).$$

2. $\partial H_0/\partial x = \partial H_0/\partial y = 0$ for $x = y = 0$.
3. The functional determinant of the matrix

$$\mathcal{H}_0 = \begin{vmatrix} \partial^2 H_0/\partial x\,\partial y & -\partial^2 H_0/\partial x^2 \\ \partial^2 H_0/\partial y^2 & -\partial^2 H_0/\partial y\,\partial x \end{vmatrix}$$

does not vanish in the domain G for $x = y = 0$*.

4. The functional determinant $\det|\partial^2 H_0/\partial p^2|$ does not vanish in the domain G for $x = y = 0$.

5. In the domain G the conditions

$$\lambda_\alpha(p) - i(\omega, k) \neq 0, \qquad \lambda_\alpha(p) + \lambda_\beta(p) - i(\omega, k) \neq 0,$$

are satisfied, where $\lambda_\alpha(p)$ and $\lambda_\beta(p)$ are arbitrary eigenvalues of the matrix $\mathscr{H}_0$ for $x = y = 0$, $\omega = \omega(p) = \partial H_0/\partial p|_{x=y=0}$, $(\omega, k) = \sum_{j=1}^{n} \omega_j k_j$, and k_j are arbitrary integers.

6. The function $H_1(x, y, p, q)$ is analytic in x, y, p, q in the domain F:

$$|x| < h, \quad |y| < h, \quad p \in G, \quad |\operatorname{Im} q| < \rho$$

and is periodic in q with period 2π $(\rho > 0)$.

Then for any $\chi > 0$ there exists an $M = M(\chi, \rho, h, G, H_0) > 0$ such that if in the domain F: $|x| < h,\ |y| < h,\ p \in G,\ |\operatorname{Im} q| < \rho$, the inequality

$$|H_1(x, y, p, q)| < M$$

holds, then there exists a decomposition $\tilde{G} = G_1 \cup G_2$ of the domain $\tilde{G} = \operatorname{Re} G$** such that G_2 is small, i.e. $\operatorname{mes} G_2 \leq \chi \operatorname{mes} \tilde{G}$, and for every $p \in G_1$ there exists a torus $T_{p\omega}$, invariant with respect to the motions of system (1), possessing the following properties:

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* This condition is superfluous if $\partial H_1/\partial x = \partial H_1/\partial y = 0$ for $x = y = 0$.

** $\tilde{G} = \operatorname{Re} G$ is the intersection of the domain G with the subspace $\operatorname{Im} p = 0$; here the domain G is assumed to be bounded.

1. The invariant tori $T_{p\omega}$ are given by the parametric equations

$$x = f_\omega(Q), \qquad y = g_\omega(Q), \qquad p = p_\omega + h_\omega(Q), \qquad q = Q + r_\omega(Q),$$

where $f_\omega(Q), g_\omega(Q), h_\omega(Q)$, and $r_\omega(Q)$ are analytic functions of Q, periodic in Q with period 2π.

2. The invariant tori T_p differ little from the unperturbed tori $x = y = 0,\ p = p_\omega$, i.e.,

$$|f_\omega(Q)| < \chi, \qquad |g_\omega(Q)| < \chi, \qquad |h_\omega(Q)| < \chi, \qquad |r_\omega(Q)| < \chi.$$

3. The motion on the torus $T_{p\omega}$ is conditionally periodic with n frequencies $\omega = (\omega_1, \ldots, \omega_n)$, i.e. $\dot{Q} = \omega$, where $\omega = \partial H_0/\partial p\big|_{x=y=0,\ p=p_\omega}$.

The proof of this theorem is based on the possibility of transforming system (1), by means of a canonical change of variables, into a Hamiltonian system with Hamiltonian function of the form

$$H = \bar{H}(X, Y, P) + H_2(X, Y, P, Q);$$

where $\partial\bar{H}/\partial X = \partial\bar{H}/\partial Y = 0$ for $X = Y = 0$, and the function $H_2(X, Y, P, Q)$ in the domain F': $|X| < h' < h,\ |Y| < h',\ P \in G' \subset G,\ |\operatorname{Im} Q| < \rho' < \rho$ $(h' > 0, \rho' > 0)$ satisfies the inequality

$$|H_2(X, Y, P, Q)| < M^{1+\frac{1}{6}}, \tag{2}$$

and the measure of the difference $G \setminus G'$ is small together with M.

For this purpose put

$$H(x, y, p, q) = \bar{H}(x, y, p) + \tilde{H}(x, y, p, q), \tag{3}$$

where

$$\bar{H}(x, y, p) = H_0(x, y, p) + \bar{H}_1(x, y, p),$$

$$\bar{H}_1(x, y, p) = (2\pi)^{-n} \oint H_1(x, y, p, q)\, dq$$

and the contour integral in the last equality is taken over the surface of the n-dimensional torus. Let, further, $x = f(p)$; $y = g(p)$ be the solution of the system of equations $\partial\bar{H}/\partial x = \partial\bar{H}/\partial y = 0$, which reduces to the solution $x = y = 0$ when $\bar{H}_1(x, y, p) \equiv 0$. We now perform in system (1) the canonical change of variables (3)

$$x = x' + f(p'), \qquad y' = y - g(p'), \qquad p = p', \qquad q' = q + \frac{\partial f}{\partial p'} y - \frac{\partial g}{\partial p'} x'$$

with generating function $V = x'y + p'q + f(p')y - g(p')x'$. As a result of the change of variables the Hamiltonian function of system (1), according to (3), takes the form

$$H(x', y', p', q') = \bar{H}(x', y', p') + \tilde{H}(x', y', p', q'),$$

where

$$\oint \tilde{H}(x', y', p', q')\, dq' = 0$$

and $\partial\bar{H}/\partial x' = \partial\bar{H}/\partial y' = 0$ for $x' = y' = 0$.

The canonical change of variables

$$x' = X + \partial S/\partial y', \qquad Y = y' + \partial S/\partial X, \qquad p' = P + \partial S/\partial q',$$

$$Q = q' + \partial S/\partial P \tag{4}$$

with generating function $V = Xy' + Pq' + S(X, y', P, q')$ brings $H(x', y', p', q')$ to the form

$$H = \bar{H}(X, Y, P) + R_1 + R_2 + R_3 + R_4 + R_5, \tag{5}$$

where

$$R_1 = \sum_{\alpha=1}^{n} \frac{\partial S}{\partial q'_\alpha} \frac{\partial}{\partial P_\alpha} H(X, y', P) + [\tilde{H}(X, y', P, q')]_N +$$

$$+ \sum_{\alpha=1}^{m} \left(\frac{\partial S}{\partial y'_\alpha} \frac{\partial}{\partial X_\alpha} \bar{H}(X, y', P) - \frac{\partial S}{\partial X_\alpha} \frac{\partial}{\partial y'_\alpha} \bar{H}(X, y', P) \right),$$

$$R_2 = \overline{H}(x', y', p') - \overline{H}(X, y', P) -$$

$$- \sum_{\alpha=1}^{m} \frac{\partial S}{\partial y'_\alpha} \frac{\partial}{\partial X_\alpha} \overline{H}(X, y', P) - \sum_{\alpha=1}^{n} \frac{\partial S}{\partial q'_\alpha} \frac{\partial}{\partial P_\alpha} \overline{H}(X, y', P),$$

$$R_3 = \overline{H}(X, y', P) - \overline{H}(X, Y, P) + \sum_{\alpha=1}^{m} \frac{\partial S}{\partial X_\alpha} \frac{\partial}{\partial y'_\alpha} \overline{H}(X, y', P),$$

$$R_4 = \widetilde{H}(X, y', P, q') - [\widetilde{H}(X, y', P, q')]_N,$$

$$R_5 = \widetilde{H}(x', y', p', q') - \widetilde{H}(X, y', P, q'),$$

$$\widetilde{H}(x', y', p', q') = \sum_{|k| \neq 0} h_k(x', y', p') e^{i(k, q')},$$

$$[\widetilde{H}(X, y', P, q')]_N = \sum_{0 < |k| < N} h_k(X, y', P) e^{i(k, q')}.$$

and the variables x', y', p', q' are expressed in terms of X, Y, P, Q according to the equalities (4).

Take the function $S(X, y', P, q')$ in the form

$$S = S_0 + S_X X + S_{y'} y' + {}^1/_2 S_{XX} X^2 + S_{Xy'} X y' + {}^1/_2 S_{y'y'} {y'}^2,$$

where the function $S_0 = S_0(P, q')$ satisfies the equation

$$(\omega, \partial S_0/\partial q') + [\widetilde{H}_0(P, q')]_N = 0, \tag{6}$$

the vectors $S_X = S_X(P, q')$ and $S_{y'} = S_{y'}(P, q')$ satisfy the system of equations

$$\begin{aligned} &\left(\omega, \frac{\partial S_X}{\partial q'}\right) + [\widetilde{H}_X(P, q')]_N = \frac{\partial^2 \overline{H}}{\partial X\, \partial y'} S_X - \frac{\partial^2 \overline{H}}{\partial X^2} S_{y'}, \\ &\left(\omega, \frac{\partial S_{y'}}{\partial q'}\right) + [\widetilde{H}_y(P, q')]_N = \frac{\partial^2 \overline{H}}{\partial {y'}^2} S_X - \frac{\partial^2 \overline{H}}{\partial y'\, \partial X} S_{y'}, \end{aligned} \tag{7}$$

and the matrices $S_{XX} = S_{XX}(P, q')$, $S_{Xy'} = S_{Xy'}(P, q')$, and $S_{y'y'} = S_{y'y'}(P, q')$ satisfy the system of matrix equations

$$\begin{aligned} &\left(\omega, \frac{\partial S_{XX}}{\partial q'}\right) + [\widetilde{H}_{XX}(P, q')]_N = S_{XX} \frac{\partial^2 \overline{H}}{\partial y' \partial X} + \frac{\partial^2 \overline{H}}{\partial X \partial y'} S_{XX} - \\ &- S_{Xy'} \frac{\partial^2 \overline{H}}{\partial X^2} - \frac{\partial^2 \overline{H}}{\partial X^2} S'_{Xy'} + \sum_{\alpha=1}^{m} \left(S_{X_\alpha} \frac{\partial}{\partial y'_\alpha} \frac{\partial^2 \overline{H}}{\partial X^2} - S_{y'_\alpha} \frac{\partial}{\partial X_\alpha} \frac{\partial^2 \overline{H}}{\partial X^2} \right); \\ &\left(\omega, \frac{\partial S_{Xy'}}{\partial q'}\right) + [\widetilde{H}_{Xy'}(P, q')]_N = S_{XX} \frac{\partial^2 \overline{H}}{\partial {y'}^2} + \frac{\partial^2 \overline{H}}{\partial X \partial y'} S_{Xy'} - \\ &- S_{Xy'} \frac{\partial^2 \overline{H}}{\partial X \partial y'} - \frac{\partial^2 \overline{H}}{\partial X^2} S_{y'y'} + \sum_{\alpha=1}^{m} \left(S_{X_\alpha} \frac{\partial}{\partial y'_\alpha} \frac{\partial^2 \overline{H}}{\partial X \partial y'} - S_{y'_\alpha} \frac{\partial}{\partial X_\alpha} \frac{\partial^2 \overline{H}}{\partial X \partial y'} \right), \\ &\left(\omega, \frac{\partial S_{y'y'}}{\partial q'}\right) + [\widetilde{H}_{y'y'}(P, q')]_N = S'_{Xy'} \frac{\partial^2 \overline{H}}{\partial {y'}^2} + \frac{\partial^2 \overline{H}}{\partial {y'}^2} S_{Xy'} - \\ &- S_{y'y'} \frac{\partial^2 \overline{H}}{\partial X \partial y'} - \frac{\partial^2 \overline{H}}{\partial y' \partial X} S_{y'y'} + \sum_{\alpha=1}^{m} \left(S_{X_\alpha} \frac{\partial}{\partial y'_\alpha} \frac{\partial^2 \overline{H}}{\partial {y'}^2} - S_{y'_\alpha} \frac{\partial}{\partial X_\alpha} \frac{\partial^2 \overline{H}}{\partial {y'}^2} \right). \end{aligned} \tag{8}$$

Here $\overline{H} = \overline{H}(X, y', P)$; $\omega = \omega(P) = \partial H/\partial P\big|_{X=y'=0}$; the matrices $\partial^2 \overline{H}/\partial X^2$, $\partial^2 \overline{H}/\partial X \partial y'$, $\partial^2 \overline{H}/\partial {y'}^2$ are taken at $X = y' = 0$; $[\widetilde{H}_0(P, q')]_N$, $[\widetilde{H}_X(P, q')]_N$, etc. denote, respectively, the value of the function $[\widetilde{H}(X, y', P, q')]_N$ and of its partial derivatives at $X = y' = 0$.

Let now in the domain $G_N^\delta \subset G - \beta$ * the conditions be fulfilled

$$|(\omega, k)| > \delta |k|^{-(n+1)}, \qquad |\lambda_\alpha - i(\omega, k)| > \delta |k|^{-(n+1)},$$

$$|\lambda_\alpha + \lambda_\beta - i(\omega, k)| > \delta |k|^{-(n+1)}$$

for all integral values $k = (k_1, \dots, k_n)$ such that

$$|k| = \sum_{j=1}^{n} |k_j| < N.$$

Then, for sufficiently small $\delta > 0$, in the domain $P \in G_N^\delta$, $|\operatorname{Im} q'| < \rho - 3\delta$, the system of equations (6), (7), (8) has an analytic solution satisfying in this domain the inequalities

$$|S_0(P,q')| < M\delta^{-(2n+3)}, \qquad |S_X(P,q'),\, S_{y'}(P,q')| < Mh^{-1}\delta^{-(m+1)(n+2)+1},$$

$$|S_{XX}(P,q'),\, S_{Xy'}(P,q'),\, S_{y'y'}(P,q')| < Mh^{-2}\delta^{-3m(n+2)+1}.$$

It follows from this that in the domain F': $|X| < h'$, $|Y| < h'$, $P \in G_N^\delta - 2\beta$, $|\operatorname{Im} Q| < \rho' = \rho - 3\gamma$, for

$$N = \frac{1}{\gamma} \ln \frac{h^2}{2M}$$

the inequalities

$$|R_1| < Mh^{-3}{h'}^3\delta^{-\nu}, \qquad |R_2, R_3, R_4, R_5| < M^2h^{-2}\delta^{-2\nu},$$

are valid, where $\nu = 3m(n+2)$. A corresponding choice of $\beta > 0$ makes it possible to make the measure of the difference $G \setminus G'$ small together with M. Putting further $M = \delta^T$, $h = \delta^\Gamma$, $h' = h^{1+1/6}$, where $T = 36\nu$, $\Gamma = 14\nu$, we obtain that in the domain F' inequality (2) is fulfilled; moreover, the partial derivatives of the function $H_2(X,Y,P,Q)$ with respect to X and Y up to and including second order will be small together with M for $X = Y = 0$. This makes it possible to apply the constructed procedure an unlimited number of times. The convergence of the process is proved analogously to how this is done in work ([2]).

In conclusion it is necessary to note that the formulated theorem, with obvious modifications, remains valid also in the case of the canonical system

$$\dot{x} = -\partial H/\partial\varphi, \qquad \dot{\varphi} = \partial H/\partial x \quad (x = (x_1, \dots, x_m),\ \varphi = (\varphi_1, \dots, \varphi_m)), \tag{9}$$

$$\dot{p} = -\partial H/\partial q, \qquad \dot{q} = \partial H/\partial p \quad (p = (p_1, \dots, p_n);\ q = (q_1, \dots, q_n))$$

with Hamiltonian function of the form

$$H = H_0(x,p) + \varepsilon(H_1(x,p,\varphi) + H_2(x,p,\varphi,q)),$$

apart from the obvious conditions of analyticity in x, p, φ, q and periodicity in q, satisfying the following conditions:

1. $\partial H_0/\partial x = \partial H_1/\partial x = \partial H_1/\partial\varphi = 0$ for $x = \varphi = 0$.
2. The functional determinants of the matrices $\partial^2 H_0/\partial p^2$, $\partial^2 H_0/\partial x^2$, $\partial^2 H_1/\partial\varphi^2$ do not vanish in the domain G for $x = \varphi = 0$.

3. The parameter ε is sufficiently small.

The finding of conditionally periodic solutions of system (9), close to the solution $x = \varphi = 0$, $p = p_\omega$ of system (9) with $H_2(x, p, \varphi, q) \equiv 0$, plays an important role in the investigation of the phenomenon of instability in Hamiltonian systems close to integrable ones ([4]).

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CITED LITERATURE

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* The domain $G - \beta \subset G$ contains those points of the domain G whose distance to the boundary of the domain G is greater than β ($\beta > 0$).

Note: Figure translations are in progress. See original paper for figures.