Corresponding Member of the USSR Academy of Sciences G. I. MARCHUK, U. M. SULTANGAZIN

1965

SovietRxiv

Abstract

Full Text

MATHEMATICAL PHYSICS

Corresponding Member of the USSR Academy of Sciences G. I. MARCHUK, U. M. SULTANGAZIN

ON THE CONVERGENCE OF THE SPLITTING METHOD FOR THE RADIATIVE TRANSFER EQUATION

In the work ([1]) an algorithm was proposed for solving the multidimensional kinetic equation of radiative transfer by means of the splitting method. In the present work we study questions of justification of this method as applied to the simplest transfer equation for plane-parallel geometry under the assumption of isotropic scattering. If φ is the radiation flux, and σ and σ_s are the cross sections for the interaction of radiation with matter, then the transfer equation takes the form

$$\mu \frac{\partial \varphi}{\partial z} + \sigma \varphi = \frac{\sigma_s}{2} \int_{-1}^{1} \varphi \, d\mu + f(z, \mu). \tag{1}$$

The solution of equation (1) is sought in the rectangle $G\{0 \leq z \leq 1,\ -1 \leq \mu \leq 1\}$, taking into account the boundary conditions

$$\varphi(0, \mu) = \xi(\mu), \qquad \text{for } \mu > 0; \tag{2}$$

$$\varphi(1, \mu) = \eta(\mu), \qquad \text{for } \mu < 0, \tag{3}$$

where $\xi(\mu)$ and $\eta(\mu)$ are prescribed functions.

Let us introduce the operators

$$\mathcal{L}_1 = \mu \frac{\partial}{\partial z}, \qquad \mathcal{L}_2 = \sigma E - \frac{\sigma_s}{2} \int_{-1}^{1} d\mu \tag{4}$$

and write equation (1) in the form

$$(\mathcal{L}_1 + \mathcal{L}_2)\varphi = f. \tag{5}$$

Construct the grid domain $G_h^l\{z_i = ih,\ \mu_j = jl,\ h = 1/N,\ l = 1/M,\ 0 \le i \le N,\ -M \le j \le M\}$ and define in it a vector $\bar{\varphi}$ of dimension $2NM$, whose components are the quantities $\varphi_{ij} = \varphi(ih, jl)$ for all $i, j \in G_h^l - \Gamma_h^l - S_h^l$, where Γ_h^l is the set of points of the grid domain G_h^l for which $i = 0,\ j = 1, 2, \dots, M;\ i = N,\ j = -1, -2, \dots, -M;\ S_h^l$ is the set for $j = 0,\ i = 1, 2, \dots, N$. Now let us compose a finite-difference approximation of the problem. For this purpose we approximate the operator $\mathcal{L}_1$ by means of a finite-difference operator of first order of accuracy with respect to h, and $\mathcal{L}_2$ by the corresponding expression using a quadrature formula. Then, taking the boundary conditions into account, we obtain a system of linear algebraic equations

$$(\Lambda_1 + \Lambda_2)\bar{\varphi} = \bar{f}, \tag{6}$$

where $\Lambda_1 = [\Lambda_1^{(1)}, \Lambda_1^{(2)}, \dots, \Lambda_1^{(2M)}]$ is a block-quasidiagonal matrix whose elements are

$$\Lambda_1^{(i)} = \begin{Vmatrix} d_i & 0 & 0 & \cdots & 0 & 0 \\ -d_i & d_i & 0 & \cdots & 0 & 0 \\ 0 & -d_i & d_i & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -d_i & d_i \end{Vmatrix}, \qquad d_i = \frac{\mu_i}{h} \quad (i = 1, 2, \dots, M);$$

$$\Lambda_1^{(2M+1-i)} = \begin{Vmatrix} d_i & -d_i & 0 & \dots & 0 & 0 \\ 0 & d_i & -d_i & \dots & 0 & 0 \\ 0 & 0 & d_i & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & d_i \end{Vmatrix} \qquad (i = 1, 2, \dots, M);$$

Λ_2 is a symmetric matrix of dimension $2NM$.

Define the following iterative process:

$$(E + \tau\Lambda_1)(E + \tau\Lambda_2)\bar{\varphi}^{k+1} = (E - \tau\Lambda_1)(E - \tau\Lambda_2)\bar{\varphi}^k + 2\tau\bar{f}; \tag{7}$$

here E is the identity matrix, and τ is a certain relaxation parameter. We implement scheme (7) in the following way:

$$\bar{\varphi}^{k+1/4} = (E - \tau\Lambda_2)\bar{\varphi}^k,$$

$$\bar{\varphi}^{k+2/4} = (E - \tau\Lambda_1)\bar{\varphi}^{k+1/4}, \tag{8}$$

$$(E + \tau\Lambda_1)\bar{\varphi}^{k+3/4} = \bar{\varphi}^{k+2/4} + 2\tau\bar{f},$$

$$(E + \tau\Lambda_2)\bar{\varphi}^{k+1} = \bar{\varphi}^{k+3/4}.$$

Introduce the vector

$$e^k = \bar{\varphi}^k - \bar{\varphi}. \tag{9}$$

Then we arrive at the following homogeneous equation:

$$e^{k+1} = Te^k, \tag{10}$$

where

$$T = (E + \tau\Lambda_2)^{-1}(E + \tau\Lambda_1)^{-1}(E - \tau\Lambda_1)(E - \tau\Lambda_2). \tag{11}$$

The matrix Λ_2 has two distinct eigenvalues: σ, σ_c. Therefore, from the symmetry of the matrix Λ_2 there follows the assertion:

Lemma 1. If $\sigma_c \neq 0$, then the matrix Λ_2 is positive definite. If $\sigma_c = 0$, then Λ_2 is a nonnegative matrix.

Lemma 2. The matrix Λ_1 is positive definite.

The assertion of Lemma 2 follows from the identity

$$(\Lambda_1\bar{\varphi}, \bar{\varphi}) = (\overline{\Lambda}_1\bar{\varphi}, \bar{\varphi}), \tag{12}$$

where

$$\overline{\Lambda}_1 = \left[\overline{\Lambda}_1^{(1)}, \overline{\Lambda}_1^{(2)}, \ldots, \overline{\Lambda}_1^{(2M)}\right].$$

The elements of the matrix Λ_1 are Jacobi matrices of the form

$$\overline{\Lambda}_1^{(i)} = \left\| \begin{matrix} d_i & -\frac{1}{2}d_i & 0 & \ldots & \ldots & 0 & 0 \\ -\frac{1}{2}d_i & d_i & -\frac{1}{2}d_i & \ldots & \ldots & 0 & 0 \\ 0 & -\frac{1}{2}d_i & d_i & \ldots & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & -\frac{1}{2}d_i & d_i & \end{matrix} \right\|. \tag{13}$$

On the basis of Lemmas 1 and 2, the following holds.

Theorem 1. For any $\tau > 0$, the iterative process (8) converges to the solution of system (6) (see [4]).

Introduce the spectral norm of a matrix:

$$\|\Lambda\| = \max_{\|\bar\varphi\|=1} |\Lambda\bar\varphi|.$$

Then the following lemmas are valid:

Lemma 3.

$$\left\|(E+\tau\Lambda_2)^{-1}(E-\tau\Lambda_2)\right\| = \max_i \left|\frac{1-\tau\lambda_i}{1+\tau\lambda_i}\right|,$$

where λ_i are the eigenvalues of the matrix Λ_2.

Lemma 4.

$$\left\|(E+\tau\Lambda_1)^{-1}(E-\tau\Lambda_1)\right\| \leq$$

$$\leq \max_i \max_j \frac{\left[1+4\tau^2 d_i^2\cos^2(\theta_j/2) - 4\tau d_i \cos^2(\theta_j/2)\right]^{1/2}}{\left[1+4\tau^2 d_i^2\cos^2(\theta_j/2) + 4\tau d_i \cos^2(\theta_j/2)\right]^{1/2}},$$

where $\theta_j = \pi j/(N+1)$ $(j = 1, 2, \dots, N)$.

Taking into account that the matrix $Q = (E+\tau\Lambda_2)T(E+\tau\Lambda_2)^{-1}$ is similar to the matrix T, and taking into consideration the estimates of the eigenvalues by Gershgorin, on the basis of Lemmas 3 and 4 we have the estimate

$$\rho(T) \leq \max_i \max_j \frac{\left[1+4\tau^2 d_i^2\cos^2(\theta_j/2) - 4\tau d_i \cos^2(\theta_j/2)\right]^{1/2}}{\left[1+4\tau^2 d_i^2\cos^2(\theta_j/2) + 4\tau d_i \cos^2(\theta_j/2)\right]^{1/2}} \max_i \left|\frac{1-\tau\lambda_i}{1+\tau\lambda_i}\right|. \tag{14}$$

The problem arises of choosing the optimal value of τ, for which the right-hand side of inequality (14) is minimized. This problem is difficult in general, since τ enters nonlinearly. However, one can indicate an approximate value of the optimal parameter τ. To this end, taking inequality (14) into account, we write

$$\rho(T) \leq \max_i \left|\frac{1-\tau\lambda_i}{1+\tau\lambda_i}\right|. \tag{15}$$

In this case the minimizing value of the parameter will be [5]

$$\tau = 1/\sqrt{\sigma\sigma_c}. \tag{16}$$

With the chosen optimal relaxation parameter, the iterative process converges very rapidly. The denominator of the majorizing progression has the order of magnitude

$$\gamma = (1 - \sqrt{\sigma_c/\sigma})/(1 + \sqrt{\sigma_c/\sigma}). \tag{17}$$

Next we consider an iterative scheme of first order accuracy with respect to τ

$$(E + \tau\Lambda_1)(E + \tau\Lambda_2)\overline{\varphi}^{k+1} = \overline{\varphi}^k + \tau\overline{f}. \tag{18}$$

Putting $\varphi^0 = 0$, with the aid of scheme (18) we obtain

$$\overline{\varphi}^n = \left[E - p^n - \tau^2(p^n + p^{n-1} + \cdots + p)\Lambda_1\Lambda_2\right]\overline{\varphi}, \tag{19}$$

where $p = (E + \tau\Lambda_2)^{-1}(E + \tau\Lambda_1)^{-1}$, $\overline{\varphi}$ is the exact solution of system (6). It is not difficult to establish the estimates

$$\|p\| \leq \frac{1}{1 + \tau\sigma_c} = \alpha, \qquad \|\Lambda_1\Lambda_2\| \leq \frac{4\sigma}{h^2}\cos^2\frac{\pi}{2(N+1)} = \beta. \tag{20}$$

Let us prescribe some number ε and require that the inequality hold:

$$\alpha^n + \tau^2(\alpha^n + \alpha^{n-1} + \cdots + \alpha)\beta \leq \varepsilon$$

or

$$\alpha^n + \tau^2\alpha\frac{1 - \alpha^n}{1 - \alpha}\beta \leq \varepsilon. \tag{21}$$

As $n \to \infty$ we have

$$\tau^2\frac{\alpha}{1 - \alpha}\beta \leq \varepsilon. \tag{22}$$

As a result we arrive at the inequality

$$\tau \leq \varepsilon\sigma_c/\beta. \tag{23}$$

Thus, if the error is fixed equal to ε, one can indicate an approximate value of the optimal parameter τ for which the prescribed accuracy is ensured.

For the case of bounded homogeneous media, it is necessary to introduce the effective capture cross section. Then, replacing σ_c in (16) by $\sigma_c + (\pi/H)^2$, we obtain

$$\tau = 1/\sqrt{\sigma(\sigma_c + (\pi/H)^2)}. \tag{16'}$$

It should be expected that the value of τ computed by formula (16′) will be more accurate. This assumption follows from elementary considerations, which make it possible to identify radiation leakage phenomenologically with radiation absorption in the medium.

The investigation carried out in the present work can be extended to the case of the radiation transport equation in spherical and cylindrical geometries, as well as to a scattering indicatrix symmetric with respect to the cosine of the scattering angle. By an appropriate choice of a difference approximation to the differential part of the operator of the kinetic equation, one can arrive at schemes of second-order accuracy with respect to h.

Computing Center
Siberian Branch of the Academy of Sciences of the USSR

Received
21 XI 1964

REFERENCES

1. G. I. Marchuk, N. N. Yanenko, DAN, **157**, No. 6 (1964).

2. G. I. Marchuk, *Methods for the Calculation of Nuclear Reactors*, 1961.

3. N. N. Yanenko, *Zhurnal vychislitel' noi matematiki i matematicheskoi fiziki*, **5**, 933 (1962).

4. J. Douglas, C. M. Pearcy, *Numerische Math.*, **5**, No. 2, 175 (1963).

5. V. Vazov, J. Forsythe, *Difference Methods for Solving Partial Differential Equations*, 1963.

Note: Figure translations are in progress. See original paper for figures.