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Abstract

Full Text

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ON HÖLDER CONTINUITY OF SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS OF NORMAL TYPE

(Presented by Academician I. N. Vekua on 8 I 1965)

I. N. Vekua stated the following hypothesis.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, with $\mathcal{Y}$ a subspace of $\mathcal{X}$, and let a linear operator H, acting in these spaces, map $\mathcal{X}$ into $\mathcal{X}$ and $\mathcal{Y}$ into $\mathcal{Y}$. Consider the equation

$$H\mu = f, \tag{1}$$

where $f \in \mathcal{Y}$. Suppose that equation (1) is solvable for every $f \in \mathcal{X}$. If equation (1) has a solution μ in the space $\mathcal{X}$, then $\mu \in \mathcal{Y}$.

In the case when the operator H has an inverse operator possessing the same properties as H, this theorem is obvious.

In the present note this assertion is proved (in a somewhat more general form) for the case when H is a singular integral operator acting in the space $\mathcal{X} \equiv L_p(E_n)$, $p > 1$, and $\mathcal{Y} \equiv L_p C_\alpha(E_n)$, $0 < \alpha < 1$. Here and below we use the notation of functional spaces adopted in the monograph of I. N. Vekua ([1]). By $C_\alpha(E_n)$ is denoted the class of functions bounded in the entire Euclidean space E_n and satisfying the Hölder condition with exponent α, $0 < \alpha \leq 1$, uniformly for any finite points x and $y \in E_n$. The norm of an element f of this space is introduced by the formula

$$C_\alpha(f, E_n) = C(f, E_n) + H_\alpha(f).$$

Consider the singular integral operator

$$H\mu = a(x)\mu(x) + \iint_{E_n} \frac{\varphi(x,\theta)}{|x-\xi|^n} \mu(\xi)\, d\xi, \tag{2}$$

where $\theta = (x-\xi)/|x-\xi|$; $a(x)$ and $\varphi(x,\theta)$ are matrices of size $l \times l$; $\mu(x)$ is a vector with l components depending on the point x of the space E_n. It is assumed that the necessary condition for the existence of the singular integral is fulfilled (see ([2])). In addition, we shall assume that the symbolic matrix $\sigma(x, y/|y|)$ (see ([2]))

of the operator (2) is unboundedly differentiable with respect to the Cartesian coordinates of the point $y \in E_n$. Let us note that, for example, boundary-value problems lead to singular equations with an operator precisely of this kind.

We shall say that H is a singular operator of class C_α^∞ (see ([3])) if its symbolic matrix $\sigma(x, y/|y|)$ admits generalized derivatives of arbitrary order s with respect to the coordinates of the point y, which belong (with respect to the coordinates of the point x) to the class $C_\alpha(E_n)$ for $|y| \geq 1$ with their constant $C_\alpha(H, s)$. The operator H is called an operator of normal type if its symbolic determinant is everywhere* different from zero.

* We assume that there exists $\lim\limits_{|x|\to\infty} \sigma(x, y/|y|)$, so that $\sigma(\infty, y/|y|)$ is defined.

Theorem. Any solution, belonging to the space $L_p(E_n)$, $p > 1$, of the system of singular integral equations of the form (1), where H is a singular operator of normal type of class C_α^∞, $0 < \alpha < 1$, belongs to the space $C_\alpha(E_n)$ with the same exponent α, provided $f \in L_p C_\alpha(E_n)$.

The proof of the theorem is based on three lemmas concerning properties of an operator of the form

$$Tf \equiv g(x) = \iint_{E_n} \frac{a(x) - a(\xi)}{|x - \xi|^n} Y(\theta) f(\xi)\, d\xi,$$

where $a(x) \in C_\alpha(E_n)$; $Y(\theta)$ is a normalized spherical function. Here $a(x)$ is an $l \times l$ matrix, and $f(x)$ and $g(x)$ are vectors with l components. Various constants depending on α, p, γ, and $C_\alpha(a)$ are denoted in the lemmas by M.

Lemma 1. Let $f \in L_p(E_n)$, $p > n/\alpha$. Then $|g(x)| \leq M L_p(f)$, $|g(x) - g(y)| \leq M L_p(f) |x - y|^\beta$, where $|x - y| \leq 1$, $\beta = (p\alpha - n)/p$.

Lemma 2. If $f \in L_p(E_n)$, $1 < p \leq n/\alpha$, then $g(x) \equiv Tf \in L_\gamma(E_n)$, where γ is arbitrary, $p < \gamma < np/(n - p\alpha)$. Moreover $L_\gamma(Tf) \leq M L_p(f)$.

Lemmas 1 and 2 were proved (for the two-dimensional case) in (1). For kernels of a more special form in the space E_n, these lemmas were established in (8). The proof in the general case is carried out analogously.

Lemma 3. If $f \in L_{pC_\beta}(E_n)$, $p > 1$, $\beta < \alpha$, $0 < \alpha < 1$, then $g(x) \equiv Tf \in C_\alpha(E_n)$. Moreover the inequality $C_\alpha(Tf) \leq M L_{pC_\beta}(f)$ holds.

Lemma 3 is proved in exactly the same way as the theorem on the Hölder continuity of the singular integral in (3).

We now pass to the proof of the theorem. By $Y_{km}(z)$, $|z| = 1$, we shall denote spherical functions of order k, and let $\{Y_{km}\}$ form a complete orthonormal basis in the space of functions summable with square on the unit sphere of the space E_n. Every operator of class C_α^∞ can be represented in the form (see (3))

$$H\mu = \sum a_{km}(x) H_{km}\mu, \tag{3}$$

where

$$H_{km}\mu = \iint_{E_n} \frac{Y_{km}(\theta)}{|x-\xi|^n}\,\mu(\xi)\,d\xi \qquad (k \geq 1).$$

By H_{00} we denote the identity operator. In addition, the functions $a_{km}(x)k^s$ for every natural s belong to the space $C_\alpha(E_n)$ with a constant independent of k and m, i.e. the estimates (see (3))

$$|a_{km}(x) - a_{km}(y)| \leq C_\alpha(H,s)k^{-s}|x-y|^\alpha, \tag{4}$$

$$|a_{km}(x)| \leq C_\alpha(H,s)k^{-s} \tag{5}$$

are valid.

By virtue of inequality (5), the series (3) converges in the norm of $L_q(E_n)$ for any q, $1 < q < \infty$.

Since the operator $H \in C_\alpha^\infty$ is of normal type, there exists for it a regularizer (see (2)) $H' \in C_\alpha^\infty$:

$$H'f = \sum b_{km}(x) H_{km} f.$$

Applying the operator H' to both sides of equation (1), we obtain

$$\mu + K\mu = H'f,$$

where

$$K\mu = \sum b_{ij}(a_{km}H_{ij} - H_{ij}a_{km})H_{km}\mu. \tag{6}$$

If $f \in L_{pC_\alpha}(E_n)$, $p > 1$, $0 < \alpha < 1$, then $H'f \in L_{pC_\alpha}(E_n)$ (see (1), theorem 1.34, and (8), theorem 1.9). Therefore it is enough to prove that the pro-

to the space $L_pC_\alpha(E_n)$ belongs every solution of the homogeneous equation

$$\mu = -K\mu. \tag{7}$$

Let $\mu \in L_p(E_n)$, $p > 1$, be a solution of equation (7), and let $p \leq n/\alpha$. Then $h_{km}(x) = H_{km}\mu \in L_p(E_n)$. Fix a sufficiently small δ so that $\varepsilon = \alpha/n - \delta > 0$. There is an integer N such that $N\varepsilon \leq 1/p < (N+1)\varepsilon$. Consider the operator

$$T_{ij}h_{km} = \iint_{E_n} \frac{a_{km}(x) - a_{km}(\xi)}{|x-\xi|^n} Y_{ij}(\theta)\, h_{km}(\xi)\, d\xi.$$

By Lemma 1, $T_{ij}h_{km} \in L_{\gamma_1}(E_n)$, where $1/\gamma_1 = 1/p - \varepsilon$. From formulas (5), (6), and (7) it follows that $\mu \in L_{\gamma_1}(E_n)$. Repeating this reasoning $2, 3, \dots, N$ times, we obtain that $\mu \in L_{\gamma_N}(E_n)$, where $1/\gamma_N = 1/p - N\varepsilon < \varepsilon$. Now $\gamma_N > 1/\varepsilon > n/\alpha$, and from Lemma 2, estimate (4), and the equation we conclude that $\mu \in C_\beta(E_n)$, where $\beta = \alpha - n/\gamma_N$. If $p > n/\alpha$, then we apply Lemma 2 immediately. Using Lemma 3, similarly to the preceding argument, we show that $\mu \in C_\alpha(E_n)$.

The theorem extends in an obvious way to the case when the singular equation (1) is considered on some sufficiently smooth manifold. We note that, using the results of the book by S. G. Mikhlin ([2]), one can weaken the smoothness requirements on the symbolic matrix $\delta(x, z)$, $|z| = 1$, with respect to the coordinates of the point z.

Sufficient conditions for Hölder continuity of any solution of the singular equation (1) were indicated in ([4]). The Hölder continuity of a solution belonging to $L_p(E_n)$, $p > 1$, of the singular integral equation (1) of normal type with any exponent $\beta < \alpha$ follows from a theorem of R. T. Seeley ([5]). This same fact (under broader assumptions) was established by T. G. Gegelia in the paper ([6]). We note that the Hölder continuity of solutions of equation (1) with the limiting exponent α for the case when the characteristic of the operator (2) does not depend on the pole follows from the results of the work of T. G. Gegelia ([7]) with the aid of the scheme given in the paper ([6]).

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REFERENCES CITED

[1] I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959.
[2] S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Moscow, 1962.
[3] A. P. Calderon, A. Zygmund, *Am. J. Math.*, **79**, 901 (1957).
[4] S. G. Mikhlin, *DAN*, **138**, No. 3 (1961).
[5] R. T. Seeley, *Am. J. Math.*, **83**, No. 2 (1961).

[6] T. G. Gegelia, *Proceedings of the Georgian Polytechnic Institute*, **1** (81), 69 (1962).
[7] T. G. Gegelia, *Proceedings of the Tbilisi Mathematical Institute*, **26**, 195 (1959).
[8] V. I. Shevchenko, Dissertation, Novosibirsk, 1964.

Note: Figure translations are in progress. See original paper for figures.