ON THE ADDITIVITY PRINCIPLE FOR CONSTRUCTING ECONOMICAL DIFFERENCE SCHEMES

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Abstract

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ON THE ADDITIVITY PRINCIPLE FOR CONSTRUCTING ECONOMICAL DIFFERENCE SCHEMES

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In all works ([1–7]) and others devoted to economical (in the number of operations) methods for the numerical solution of multidimensional problems of mathematical physics, the representability of a multidimensional elliptic operator in the form of a sum of one-dimensional operators is used in one form or another. In ([5–7]), for equations and systems of equations of parabolic and hyperbolic types, locally one-dimensional schemes (l.o.s.) were studied, constructed on the basis of the principle of modeling a multidimensional differential equation by means of a system of one-dimensional equations, with subsequent application, for the solution of each of the one-dimensional equations, of the simplest difference schemes. Below this principle of constructing locally one-dimensional schemes is applied to an abstract Cauchy problem in a Banach space (the additivity principle).

1. Let a linear (unbounded) operator $A(t)$, depending on the parameter $t \in [0, T]$, be defined in a Banach space H, with a domain of definition $D(A)$ dense in H and independent of t, and with range $\Delta(A) \subset H$. Consider the abstract Cauchy problem

$$du/dt + A(t)u = f(t), \qquad 0 \le t \le T, \qquad u(0) = u_0, \quad u_0 \in D(A), \tag{1}$$

where the derivative is understood in the strong sense; $u = u(t) \in H$ (the desired function) and $f = f(t) \in H$ (the prescribed function) are defined on the interval $0 \le t \le T$. The boundary conditions are taken into account by the requirement $u = u(t) \in D(A)$. We shall be interested in the case when A is a sum of linear operators A_α with the same domains of definition and ranges as the operator A:

$$A(t) = \sum_{\alpha=1}^{m} A_\alpha(t). \tag{2}$$

We shall agree to call $A_\alpha(t)$ one-dimensional operators, and the corresponding Cauchy problems (1_α) with $A = A_\alpha$ one-dimensional problems. We shall assume that problem (1) is uniformly well posed.

2. Introduce the mesh $\omega_\tau = \{t^j = j\tau;\ j = 0, 1, \dots, [T/\tau]\}$ with step τ. The "multidimensional" problem (1) will be modeled (approximated) by means of m one-dimensional problems.

On each interval $[t^j, t^{j+1}]$, instead of (1), we shall successively solve the system of one-dimensional differential equations

$$dv_{(\alpha)}/dt + A_\alpha(t) v_{(\alpha)}(t) = f_\alpha(t),$$

$$t \in (t^j, t^{j+1}], \qquad j = 0, 1, \dots; \qquad \alpha = 1, 2, \dots, m, \tag{3}$$

with initial conditions

$$v_{(\alpha)}(t^j) = v_{(\alpha-1)}(t^{j+1}), \quad \alpha > 1; \qquad v_{(1)}(t^j) = v_{(m)}(t^j) = v(t^j);$$

$$j \geq 0, \qquad v(0) = u_0. \tag{4}$$

By the solution of problem (3)–(4) on the mesh ω_τ we shall mean the function

$$v = v^{j+1} = v_{(m)}(t^{j+1}).$$

The functions $f_\alpha = f_\alpha(t, \tau)$ belong to the domain of definition of $f(t)$ in H, depend, generally speaking, on τ, and satisfy the normalization condition

$$\left\| \sum_{\alpha=1}^{m} f_\alpha - f(t) \right\| = O(\tau^{k_0}), \quad k_0 > 0,$$

where $\|\cdot\|$ is the norm in H. In (5) we assumed that

$$\sum_{\alpha=1}^{m} f_\alpha = f.$$

The functions $v_{(1)}^{j+1}, \dots, v_{(m)}^{j+1}$ may be interpreted as successive approximations to the solution of problem (1), the number of which m is equal to the number of terms in the sum (2).

3. Let $U(t,t')$ be the resolving operator of equation (1) for $0 \le t' \le t \le T$; let $U_\alpha(t,t')$ be the resolving operator of the one-dimensional equation (3_α) with number α. Replacing equation (1) by the locally one-dimensional system (3) means that $U(t,t')$ is approximated by the product

$$\widetilde{U}(t,t') = U_m(t,t') \dots U_1(t,t'),$$

so that $U \approx \widetilde{U}$. Thus the proposed method of one-dimensional modeling of problem (1) corresponds to an approximate factorization, or splitting, of the resolving operator $U(t,t')$ into one-dimensional operators. If $\widetilde{U} = U$, then for $f = 0$ we have $v^j = w^j$ for all $j = 0, 1, \dots$. If $j \ne 0$, then v^j coincides with w^j on the grid ω_t only under a special choice of the right-hand sides f_α, namely: $f_\alpha = 0$ for $\alpha = 1, 2, \dots, m-1$,

$$f_m(\theta) = w_{(m-1)}(t^{j+1}, \theta), \qquad t^j \le \theta \le t^{j+1},$$

where $w_{m-1}(t,\theta)$ is the solution of the problem

$$\frac{dw_{(\alpha)}}{dt}(t,\theta) + A_\alpha(t) w_{(\alpha)}(t,\theta) = 0, \qquad t^j \le \theta \le t \le t^{j+1},$$

$$w_{(\alpha)}(\theta,\theta) = w_{(\alpha-1)}(t^{j+1}, \theta), \qquad \alpha = 2, \dots, m-1,$$

$$w_{(1)}(\theta,\theta) = f(\theta).$$

Let us give the simplest examples when $\widetilde{U} = U$: 1) the heat-conduction equation with coefficients allowing separation of variables in a parallelepiped ($0 \le x_\alpha \le l_\alpha$, $\alpha = 1, 2, \dots, m$) with homogeneous boundary conditions; 2) a first-order hyperbolic-type equation with constant coefficients, etc. (V. I. Gol' din, N. N. Kalitkin). In general, if $A_\alpha(t_1) A_\beta(t_2) = A_\beta(t_2) A_\alpha(t_1)$, $\beta \ne \alpha$, $t_1, t_2 \in [0, T]$, then $U = \widetilde{U}$.

4. We shall say that problem (3), (4) is uniformly well-posed if there exist resolving operators $U_\alpha(t,t')$ for all $0 \le t' \le t \le T$, $\alpha = 1, 2, \dots, m$, strongly continuous in the aggregate t, t', and

$$\|U_\alpha(t,t')\| \le 1 + M_D(t - t'),$$

where $M_D = \text{const} > 0$ does not depend on τ.

Theorem 1. *Let problem (3), (4) be uniformly well-posed and let the solution $u = u(t)$ of problem (1) satisfy the conditions: 1)*

$$\left\|(U_\alpha(t,t')-E)\frac{du}{dt}\right\| \le \rho_1(\tau),$$

2)

$$\|(U_\alpha(t,t')-E)\psi_\alpha\| \le \rho_2(\tau), \qquad \psi_\alpha(t) = f_\alpha(t) - A_\alpha(t)u - \frac{du}{dt},$$

3)

$$\Psi_\alpha(t) = \int_{t_j}^{t} U_\alpha(t,\theta)\psi_\alpha(\theta)\,d\theta, \quad t \ge t_j,$$

is the solution of the inhomogeneous equation (3_α) *with number* α*, with right-hand side* $\psi_\alpha(t)$ *and initial condition* $\psi_\alpha(t_j)=0$*,* $\alpha=1,\dots,m$*. Then the solution of problem (3), (4) converges uniformly in* t *to the solution of problem*

$$\|v^j - u^j\| \le \rho_3(\tau), \qquad j=1,2,\dots$$

(here $\rho_k(\tau)\to 0$*,* $k=1,2,3$*, uniformly in* t*).*

5. For solving problem (3)–(4), depending on the concrete form of A_α, one may use various suitable methods (for example, seek an exact expression for $v_{(\alpha)}$, use the method of characteristics, the method of straight lines, the method of finite differences, etc.).

Following (8), we shall seek an approximate solution of problem (3), (4) in a Banach space H_N (for example, in the space of functions defined on a grid ω_N in some domain of p-dimensional space $x=(x_1,\dots,x_p)$). Let P_N be a linear operator projecting H onto H_N ($P_{Nu}=u_N\in H_N$, if $u\in H$) and $\lim_{N\to\infty}\|P_{Nu}\|_N=\|u\|$ for every

$u\in H$, where $\|\cdot\|_N$ is the norm in H_N, $\|\cdot\|$ is the norm in H.

Replacing each one-dimensional equation (3_α) of number α by a difference scheme in H_N, we obtain a L.o.s. approximating the original problem (1). We shall consider two-level schemes. To equation (3_α) there corresponds in H_N a two-level scheme $R_\alpha^N y_{(\alpha)} = S_\alpha^N y_{(\alpha)} + \tau\varphi_\alpha$, where $y_{(\alpha)}=y_{(\alpha)}^{j+1}$, $y_{(\alpha)}=y_{(\alpha)}^{j}$, and R_α^N and S_α^N are linear operators depending on t,τ,N and mapping H_N into itself (see (8)). Taking into account the initial conditions $y_{(\alpha)}=y_{(\alpha-1)}$ for $\alpha>1$ and $y_{(1)}=y_{(0)}y=y^j$, we obtain the L.o.s. in the form:

$$R_\alpha^N y_{(\alpha)} = S_\alpha^N y_{(\alpha-1)} + \tau\varphi_\alpha, \quad \alpha=1,2,\dots,m,\ t\in\omega_\tau, \quad y_{(0)}=y,$$
$$y(0)=P_{Nu}0. \tag{5}$$

The solution of this problem is the grid function $y = y^{j+1} = y^{j+1}_{(m)}$.

6. Problem (5) is posed correctly (the L.o.s. (5) is correct) if, for sufficiently large $N \geq N_0$ and sufficiently small $\tau \leq \tau_0$, its solution $y = y_{(m)}(N, \tau; t)$ for $t \in \omega_\tau$ exists for arbitrary φ_α and $y(0)$ from H_N and depends uniformly with respect to N, τ continuously on φ_α and $y(0)$, $\alpha = 1, 2, \dots, m$ (see (8)), so that, for example,

$$\max_{\omega_\tau} \|y(t)\|_N \leq M_1 \|y(0)\|_N + M_2 \max_{\alpha, \omega_\tau} \|\varphi_\alpha(t)\|_N. \tag{6}$$

Here and below M_k $(k = 1, 2, \dots)$ are positive constants independent of N and τ. For correctness of the L.o.s. (5) it is sufficient that the inverse operator $C_\alpha = (R_\alpha^N)^{-1}$ exist and that the conditions

$$\|B_m(t) \dots B_1(t)\|_N \leq 1 + M_3\tau, \quad \|B_m(t) \dots B_{s+1}(t) C_s(t)\| \leq M_4,$$
$$\|C_m(t)\|_N \leq M_5, \quad t \in \omega_\tau, \tag{7}$$

be satisfied, where $B_\alpha = (R_\alpha^N)^{-1} S_\alpha^N;\ s = 1, 2, \dots, m-1$.

Theorem 2. *Let the conditions of Theorem 1 be satisfied, let the L.o.s. (5) be correct, and let each of the one-dimensional schemes* (5_α) *of number* α *approximate the corresponding equation* (3_α) *on its solution* $v_{(\alpha)}(t)$*. Then the solution of problem (5) converges uniformly in* t *to the solution of problem (1) as* $\tau \to 0$ *and* $N \to \infty$*:*

$$\lim_{N\to\infty,\ \tau\to 0} \|y - P_{Nu}\|_N = 0$$

for $t \in \omega_\tau$.

We do not dwell here on the question of the order of accuracy of the L.o.s. (see [5–7,2]). We note that, by passing successively from (1) to (3), (4), and (5), one can strengthen some estimates (5-7). If the condition $\tilde{U} = U$ is fulfilled and, consequently, $v^j = u^j$, then for the solution of the one-dimensional problems (3_α) it is expedient to use schemes of order of accuracy higher than $O(\tau)$, since $y^j - P^j_{Nu} = y^j - P^j_{Nv}$ (for example, a scheme $O(h^4 + \tau^2)$ for the heat-conduction equation with allowance for the method of computing $f_m(t)$ indicated above, etc.).

7. Let H_N be a Euclidean space with scalar product $(y, z)_N$ and norm $\|z\|_N = \sqrt{(z, z)_N}$. Consider the family of two-level L.o.s. of the form

$$\frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau} + A_\alpha^N (\sigma_\alpha y_{(\alpha)} + (1 - \sigma_\alpha) y_{(\alpha-1)}) = \varphi_\alpha,$$
$$\alpha = 1, \dots, m,\ t \in \omega_\tau,\ y(0) = P_{Nu}0, \tag{8}$$

where A_α^N is a linear operator whose domain of definition and range coincide with H_N. For correctness of the L.o.s. (8) one of three groups of conditions is sufficient for $t \in \omega_\tau$ (cf. (8)):

Theorem 3. *If $\sigma_\alpha \geq 0.5$, $A_\alpha^N(t)$ is semibounded from below, i.e.*

$$(A_\alpha^N(t)y, y)_N \geq -c_1\|y\|_N^2$$

for any $y \in H_N$, and $\tau \leq \tau_0(c_1)$ is sufficiently small, then the L.o.s. is correct.

Theorem 4. *If $\sigma_\alpha \geq 0.5$, $(A_\alpha^{Ny}, y)_N \geq 0$, $y \in H_N$, $\tau > 0$ is arbitrary, then the L.o.s. is correct.*

Theorem 5. If $\sigma_\alpha \geqslant \sigma_\varepsilon = 0.5 - (1-\varepsilon)/\tau\|A_\alpha^N\|_N$, $(A_\alpha^N y, y)_N \geqslant 0$, A_α^N is a finite-dimensional self-adjoint operator, i.e. $(A_\alpha^N y, v)_N = (y, A_\alpha^N v)_N$, $\varepsilon \in (0,1]$, then the l.o.s. is correct.

8. Changing f_α, the numbering of A_α in (2), the number of terms in (2), etc., we obtain an innumerable set of systems (3)–(4). In particular, putting

$$A = \sum_{\alpha=1}^{m} A_\alpha = \sum_{\alpha=1}^{m} A'_\alpha, \quad A'_\alpha = \tfrac{1}{2}A_\alpha,\ \alpha \leqslant m, \quad A'_\alpha = \tfrac{1}{2}A_{2m+1-\alpha},\ \alpha > m,$$

we obtain the "symmetric" system (3), (4), which in a number of cases has accuracy $O(\tau^2)$ ($\|v^j - u^j\| = O(\tau^2)$). For $m = 2$ we have $A'_1 = \frac{1}{2}A_1$, $A'_2 = \frac{1}{2}A_2$, $A'_3 = \frac{1}{2}A_2$, $A'_4 = \frac{1}{2}A_1$ (the scheme $\frac{1}{2}A_1 \to \frac{1}{2}A_2 \to \frac{1}{2}A_2 \to \frac{1}{2}A_1$). Choosing in (8) $\sigma_1 = 0$, $\sigma_2 = 1$, $\sigma_3 = 0$, $\sigma_4 = 1$, we obtain a generalization of scheme (1), as may be verified after eliminating $y_{(1)}$, $y_{(3)}$. (This was also pointed out by I. V. Fryazinov.) Thus, scheme (1) is an l.o.s. One may also use an l.o.s. with $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = \sigma_4 = 1$. The indicated symmetrization makes it possible to obtain second-order accuracy in τ.

It is possible to use three-layer schemes (for $m = 2$ this is obvious), and also combinations of two-layer and three-layer schemes. Schemes (3) for the homogeneous ($f = 0$) heat equation are more naturally interpreted as l.o.s., and not as splitting schemes.

9. In (5) a somewhat different method of one-dimensional modeling of problem (1) was proposed. The system considered was

$$\frac{1}{m}\frac{dv_{(\alpha)}}{dt} + A_\alpha(t)v_{(\alpha)} = f_\alpha(t), \quad t \in (t_{\alpha-1}^j, t_\alpha^j), \quad t_\alpha^j = t^j + \frac{\alpha}{m}\tau, \quad \alpha = 1, 2, \dots, m,$$

$$v_{(\alpha)}(t_{\alpha-1}^j) = v_{(\alpha-1)}(t_{\alpha-1}^j), \quad \alpha > 1, \quad v_{(1)}(t^j) = v(t^j) = v_{(m)}(t^j),$$

where $v(t^{j+1}) = v_{(m)}(t^{j+1})$. If $f_\alpha = 0$ and A_α do not depend on t, then this problem is equivalent to problem (3), (4). In the general case their solutions differ by $O(\tau)$. An analogue of Theorem 1 holds (see also (10)).

Hence the same l.o.s. as before are obtained (see (9)).

10. The case

$$A = \sum_{\alpha,\beta=1}^{p} A_{\alpha\beta}$$

has also been studied. System (3) has the form ($m = 2p$)

$$\frac{dv_{(\alpha)}}{dt} + \sum_{\beta=1}^{\alpha} A^-_{\alpha\beta}(t) v_{(\beta)} = f_\alpha, \quad \alpha \leqslant p; \qquad \frac{dv_{(\alpha)}}{dt} + \sum_{\beta=p}^{2p+1-\alpha} A^+_{2p+1-\alpha,\beta}(t) v_{(\beta)} = f_\alpha,$$

$$\alpha > p,$$

with the initial conditions (4); here $(A^-_{\alpha\beta})$ and $A^+_{\alpha\beta}$ are triangular matrix-operators ($A^-_{\alpha\beta} = 0$ for $\beta > \alpha$, $A^+_{\alpha\beta} = 0$ for $\beta < \alpha$, $A^-_{\alpha\beta} + A^+_{\alpha\beta} = A_{\alpha\beta}$). The passage to an l.o.s. is evident. Such a method for constructing l.o.s. was used in (6) for a system of multidimensional parabolic equations with mixed derivatives.

11. Locally one-dimensional schemes were studied for quasilinear equations, and also for equations of second order (see (7)):

$$\frac{d^2u}{dt^2} + A(t)u = f(t), \quad u(0) = u_0, \quad \frac{du}{dt}(0) = \bar{u}_0, \quad A = \sum_{\alpha=1}^{m} A_\alpha, \quad A = \sum_{\alpha=1}^{m} (A_\alpha + A^*_\alpha),$$

where A^*_α is the operator adjoint to A_α.

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Note: Figure translations are in progress. See original paper for figures.