N. Lasnev

1.** The main result of this note is

1965

SovietRxiv

Abstract

Full Text

N. Lasnev

On Continuous Decompositions and Closed Mappings of Metric Spaces

(Presented by Academician P. S. Aleksandrov, 13 IV 1965)

1. The main result of this note is

Theorem 1*. If $\mathfrak{A} = \{A\}$ is a continuous decomposition** of a metric space S into closed sets A, then all these A, with the exception of some σ-discrete subfamily***, have empty kernel and are compact.

First of all we shall prove the following proposition:

Lemma**.** If the closed sets

$$\Phi_1, \Phi_2, \dots, \Phi_i, \dots \tag{1}$$

of a metric space X converge***** in X to a set Φ_0, which does not meet any of the sets (1), then the set

$$Q = \Phi_0 \cap \left[\bigcup_{i=1}^{\infty} \Phi_i\right]$$

is compact.

The proof is by contradiction: suppose that in Q there exists a divergent sequence of points $\{x_n\}$. Since $x_n \in [\bigcup_{i=1}^{\infty} \Phi_i]$, it is easy to construct, for $n = 1, 2, \dots$, points $y_n \in \Phi_{i_n}$ such that $\rho(y_n, x_n) < 1/2^n$ and $i_1 < i_2 < i_3 < \cdots$. Then the sequence $\{y_n\}$ also diverges, i.e. the set $\{y_n\}$ is closed, and its complement forms a neighborhood $O\Phi_0$ of the set Φ_0 containing none of the sets Φ_i, which contradicts the convergence $\{\Phi_i\} \to \Phi_0$.

We turn to the proof of Theorem 1. In every spherical neighborhood $O(A, 1/2^n)$ take a marked neighborhood, which we denote by O_A^n (here A runs through the whole system $\mathfrak{A} = \{A\}$, and n through all natural numbers). Denote by γ_n the system of all those A which, for the given n, are contained in no $O_{A'}^n$ with $A' \neq A$. The system γ_n, for every n, is conservative (and, consequently, being disjoint, discrete). Indeed, let $\delta \subseteq \gamma_n$, $\bigcup_{A\in\delta} A = B$, and $x \in [B]$. The point x is contained in some $A_x \in \mathfrak{A}$, and $O_{A_x}^n \cap B \neq \Lambda$; hence $O_{A_x}^n$ meets some $A \in \delta \subseteq \gamma_n$, and therefore $O_{A_x}^n \supset A$, and, by the definition of the system γ_n, necessarily

$A = A_x$, i.e. $A_x \in \delta$, whereby conservativeness, and hence discreteness as well, of the system γ_n are proved. It remains to prove that

---

* Compare this result with Theorem 6 of the work ([1]).

** A decomposition $\mathfrak{A} = \{A\}$ of a topological space X into disjoint closed sets A is called continuous (P. S. Aleksandrov) if every neighborhood OA_0 of any element A_0 of this decomposition $\mathfrak{A}$ contains a "marked" neighborhood $O'A_0$, i.e. a neighborhood that is the sum of some $A \in \mathfrak{A}$.

*** A family of closed sets of a given space is called discrete if every point of the space has a neighborhood meeting no more than one element of the given family; a σ-discrete system is a system which is the union of a countable number of discrete systems.

**** We note that the lemma is true for any paracompacta; moreover, in this case the set Q turns out to be bicompact.

***** In the sense of V. I. Ponomarev' s χ-topology (i.e. every neighborhood $O\Phi_0$ contains all Φ_i, starting from some index).

every $A_0 \in \mathfrak{A}$ which is not an element of the system $\gamma = \bigcup_{n=1}^{\infty} \gamma_n$ is a compactum without interior points. But the condition $A_0 \notin \gamma$ means that for every n there exists an $A_n \in \mathfrak{A}$ such that $O^n_{A_n} \cap A_0 \neq \Lambda$, i.e. $A_0 \subset O^n_{A_n}$. From the definition of $O^n_{A_n}$ it follows that for every point $y \in A_0$ we have $\rho(y, A_n) < 1/2^n$, and hence

$$A_0 \subset \left[\bigcup_{n=1}^{\infty} A_n\right].$$

By the continuity of the decomposition $\mathfrak{A} = \{A\}$, it follows that $\{A_n\} \to A_0$, and then, in view of the lemma, the set

$$A_0 \text{ (coinciding with the set } A_0 \cap \left[\bigcup_{n=1}^{\infty} A_n\right] \text{)}$$

is a compactum. From the obvious equality*

$$IA_0 \cap [X \setminus A_0] = \Lambda$$

it follows that

$$IA_0 \cap \left[\bigcup_{n=1}^{\infty} A_n\right] = \Lambda,$$

so that the presence in the set A_0 of a nonempty kernel would contradict the inclusion

$$A_0 \subset \left[\bigcup_{n=1}^{\infty} A_n\right].$$

The theorem is proved.

2. We shall derive some consequences from the theorem just proved. Let $f : S \to X$ be a closed mapping of a metric space S onto a T_1-space X. Then the decomposition of the space S into the sets $f^{-1}x$ (where $x \in X$) is continuous. By Stone' s theorem (2), the space X has countable character at all points $x \in X$ for which the boundary of the set $f^{-1}x$ is a compactum. Therefore, from Theorem 1 and from the fact that under a closed mapping f a discrete system of sets $f^{-1}x \subset S$ goes over into a discrete set of points $x \in X$, it follows

Theorem 2. *If a T_1-space X is the image of a metric space S under a closed mapping f, then X, at all its points except the points of some σ-discrete set Z, satisfies the first axiom of countability.*

Denote by $Y \subset X$ the set of all points of the space X satisfying the first axiom of countability. From Stone' s theorem (2) it follows that Y is a metrizable space.

To estimate the cardinality of the set Z, note that it is equal to the cardinality of a certain σ-discrete family of sets $f^{-1}x$ lying in the metric space, and that the cardinality of every discrete family of sets in a metric (indeed in any collectively normal) space S does not exceed the weight of this space. Thus we have:

Theorem 3.** *If a T_1-space X is the image of a metric space of weight τ under some closed mapping, then $X = Y \cup Z$, where Y is a metrizable subspace of the space X, and the set Z is σ-discrete and has cardinality $\leq \tau$.*

3. From what has been proved it follows that points of noncountable character are situated in a closed image of a metric space comparatively sparsely. Nevertheless one can construct an example of a countable regular space X which is a closed image of a metric space and which at no point satisfies the first axiom of countability. To construct the space X, denote by P the set of all dyadic rational points of the interval $(0; 1)$, and by Q the set of all its irrational points. Consider the product $P \times Q$. Each number $x \in P$ is represented uniquely in the form $m/2^n$, where m is odd. In this case we shall say that x has rank n. The set of all points $x \in P$ of rank $\leq n$ partitions the set Q into 2^n pairwise disjoint sets

$$\Delta_1^n, \Delta_2^n, \dots, \Delta_{2^n}^n :$$

each Δ_i^n is the set of all irrational numbers lying between $(i-1)/2^n$ and $i/2^n$. Finally, denoting by Δ_{ij}^n the set $(j/2^n) \times \Delta_i^n$, we obtain a decomposition $\mathfrak{A}$ of the space

---

* IA_0 denotes the open kernel of the set A_0.

** This theorem gives an answer to a question of A. V. Arhangel' skii from (1), p. 746.

$P \times Q$ into disjoint closed sets

$$\Delta_{ij}^{n}, \qquad n = 1, 2, \dots; \quad i = 1, 2, \dots, 2^n; \quad j = 1, 2, \dots, 2^{n-1}.$$

We shall prove that the decomposition $\mathfrak{A}$ is continuous. Consider an arbitrary element $\Delta_{i_0 j_0}^{n_0}$ and an arbitrary neighborhood $O\Delta_{i_0 j_0}^{n_0}$ of it. Let $x_0 = (p_0, q_0)$ be some point of $\Delta_{i_0 j_0}^{n_0}$, and let ε be its distance to the set $(P \times Q) \setminus O\Delta_{i_0 j_0}^{n_0}$. Choose n_1 so that $1/2^{n_1} < \varepsilon/\sqrt{2}$. Choose the number i_1 so that $(i_1 - 1)/2^{n_1} < q_0 < i_1/2^{n_1}$, and consider the set

$$\Delta_{x_0} = \left(p_0 - \frac{1}{2^{n_1}},\, p_0 + \frac{1}{2^{n_1}}\right) \times \Delta_{i_1}^{n_1},$$

where $(p_0 - 1/2^{n_1},\, p_0 + 1/2^{n_1})$ is an interval in the set P of dyadic-rational numbers. Obviously, the set Δ_{x_0} is open, contains x_0, lies in $O\Delta_{i_0 j_0}^{n_0}$, and includes all elements of the decomposition Δ_{ij}^{n} that intersect it, except, perhaps, $\Delta_{i_0 j_0}^{n_0}$. Taking the union of the sets Δ_{x_0} over all $x_0 \in \Delta_{i_0 j_0}^{n_0}$, we obtain the indicated neighborhood of the element $\Delta_{i_0 j_0}^{n_0}$, inscribed in $O\Delta_{i_0 j_0}^{n_0}$. This proves the continuity of the decomposition $\mathfrak{A}$.

Obviously, every element of the decomposition $\mathfrak{A}$ coincides with its boundary and is not compact. Consequently, by Stone' s theorem ([2]), the space X of the decomposition $\mathfrak{A}$ has countable character at none of its points, as was required to prove. Finally, let us prove the assertion:

Theorem 4. *Let $f : X \to Y$ be a closed mapping of a paracompact X onto a Fréchet-Urysohn* space Y. Then there exists in X a closed set P such that $fP = Y$, and the mapping f is irreducible on the set P.*

Proof. Consider the set Z of all isolated points of the space Y, and for each $y \in Z$ choose one point from $f^{-1}y$, obtaining in this way a set $Q \subset X$, $fQ = Z$. Now consider the closed subspace

$$S = Q \cup (f^{-1}(Y \setminus Z))$$

of the space X. Suppose that f is not an irreducible mapping on S. Then there exists an open set U_1 in S such that $f(S \setminus U_1) = Y$. Suppose that for each transfinite number $\alpha < \beta$ an open set U_α has been constructed in such a way that $f(S \setminus U_\alpha) = Y$ and $U_{\alpha_1} \subset U_{\alpha_2}$ for $\alpha_1 < \alpha_2$. If β is a limit number, put $U_\beta = \bigcup_{\alpha<\beta} U_\alpha$. We shall show that $f(S \setminus U_\beta) = Y$. Suppose this is not so. Then some inverse image $f^{-1}y_0$ is contained in U_β, but, by construction, no U_α contains any inverse image $f^{-1}y$. Consider some sequence of sets $f^{-1}y_1, \dots, f^{-1}y_n, \dots$ converging to $f^{-1}y_0$. By the lemma, the set

$$T = f^{-1}y_0 \cap \left[\bigcup_{n=1}^{\infty} f^{-1}y_n\right]$$

is bicompact. Then T is contained in some set U_α, $\alpha < \beta$. But the sequence $\{f^{-1}y_n\}$ also converges to the set T. Consequently, beginning with some n, all $f^{-1}y_n$ are contained in $U_\alpha \supset T$, contrary to the fact that $f(S \setminus U_\alpha) = Y$. Now let $\beta = \beta_1 + 1$. In this case we choose arbitrarily such a U_β that $f(S \setminus U_\beta) = Y$ and $U_\beta \supset U_{\beta_1}$. This induction terminates at some transfinite stage; as a result, one obtains a set U such that the mapping f on the set $S \setminus U$ is irreducible. The theorem is proved.

The work was written under the supervision of V. I. Ponomarev, to whom I express my gratitude.

Moscow State University
named after M. V. Lomonosov

Received
25 III 1965

REFERENCES

1. A. V. Arkhangel' skii, DAN, **153**, No. 4, 743 (1963).

2. A. H. Stone, Proc. Am. Math. Soc., **7**, 690 (1956).

3. V. I. Ponomarev, Matem. sborn., **48**, 211 (1959).

* Y is a Fréchet–Urysohn space if for every set $M \subseteq Y$ and point $y_0 \in [M]$ there exists a countable sequence $\{y_n\}$ of points of M converging to the point y_0.

Note: Figure translations are in progress. See original paper for figures.