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Abstract

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ON ESTIMATES IN THE THEORY OF REGULAR FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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1. The present note contains some new investigations by the author* concerning estimates in the theory of regular functions of several complex variables. In addition, a refinement is given here of a theorem of I. A. Aleksandrov ((3), Theorem 5) on estimates of Taylor coefficients of typically real functions of several complex variables. In fact, the investigations are carried out for the case of two complex variables, since in the case of n complex variables they are carried out in a completely analogous way.

2. Let D be a bounded complete bicircular domain with center at the point $(0,0)$; let K be a bounded complete circular domain with center at the point $(0,0)$. It is known (4) that every function $F(w,z)$, regular in D, besides its representation in D by the series

$$F(w,z)=\sum_{m,n=0}^{\infty} a_{mn} w^m z^n,$$

can be represented in this domain by the diagonal series

$$F(w,z)=\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l\right). \tag{1}$$

3. Theorem 1. If in the domain D the function (1) is regular and $\operatorname{Re} F(w,z) \geqslant 0$, then for $k>0$ the sharp estimates hold

$$A_k(D) \leqslant 4(\operatorname{Re} a_{00})^2, \qquad B_k(D) \leqslant 2\operatorname{Re} a_{00},$$

where

$$A_k(D) \equiv \sup_{(w,z)\in D} \sum_{l=0}^{k} |a_{k-l,l}|^2 |w|^{2(k-l)} |z|^{2l},$$

$$B_k(D) \equiv \sup_{(w,z)\in D} \left|\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l\right|.$$

Proof. Let (w_0, z_0) be an arbitrary point of the domain D. Since the domain D is complete, for $|\zeta| < 1,\ 0 \le t \le 2\pi$ the points $(\zeta w_0,\ \zeta z_0 e^{-it}) \in D$. Therefore, for any fixed value of $t,\ 0 \le t \le 2\pi$, in the disk $|\zeta| < 1$ the function

$$F(\zeta w_0, \zeta z_0 e^{-it}) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w_0^{k-l} z_0^l e^{-ilt} \right) \zeta^k$$

is regular and $\operatorname{Re} F(\zeta w_0, \zeta z_0 e^{-it}) \geqslant 0$. Consequently (5), for $0 \le t \le 2\pi$ we have

$$\left| \sum_{l=0}^{k} a_{k-l,l} w_0^{k-l} z_0^l e^{-ilt} \right| \leqslant 2 \operatorname{Re} a_{00} \quad (k > 0). \tag{2}$$

* Previous investigations of the author, see, for example, (1, 2).

Squaring both parts of inequality (2) and integrating the resulting inequality with respect to t from 0 to 2π, we obtain the estimate leading to $A_k(D) \le 4(\operatorname{Re} a_{00})^2\ (k > 0)$. For $t = 0$, inequality (2) leads to the estimate $B_k(D) \le 2 \operatorname{Re} a_{00}\ (k > 0)$. Finally, the estimates obtained are sharp, since there exist functions for which they are attained.

Corollary 1. If in the domain D the function (1) is regular and $\operatorname{Re} F(w, z) \le U$, then for $k > 0$ the sharp estimates hold

$$A_k(D) \le 4(U - \operatorname{Re} a_{00})^2, \quad B_k(D) \le 2(U - \operatorname{Re} a_{00}).$$

Using the known estimates in the case of one variable (([6], Theorem 10; ([7], Theorem A)), as well as Theorem 1, one establishes Theorems 2 and 3.

Theorem 2. If in the domain D the function (1), where a_{00} is prescribed, is regular and $|F(w, z)| \le 1$, then for $k > 0$ the sharp estimates hold

$$A_k(D) \le (1 - |a_{00}|^2)^2, \quad B_k(D) \le 1 - |a_{00}|^2.$$

Theorem 3. If the function (1) $(a_{00} = 0)$ is regular in the domain D and satisfies the condition $F(w_1, z_1)F(w_2, z_2) \ne 1$, for arbitrary (w_1, z_1) and (w_2, z_2) from D, then for $k > 0$ the sharp estimates hold

$$A_k(D) \le 1, \quad B_k(D) \le 1.$$

Remark 1. From the first estimates of Theorems 1-3 and Corollary 1 there follow the previously established ([3,8a]) estimates of the Taylor coefficients of the indicated functions.*

Remark 2. The second estimates in Theorems 1-3 and in Corollary 1, with $B_k(D)$ replaced by $B_k(K)$, remain valid also under the conditions of Theorems 1-3 and Corollary 1 in the case of the domain K.

I. A. Aleksandrov obtained the following proposition (see ([3], Theorem 5)).

Proposition 1 (I. A. Aleksandrov). Suppose that the function

$$F(z_1, \dots, z_n) = \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1\dots k_n} z_1^{k_1} \cdots z_n^{k_n}$$

is typically real in the complete multicircular domain D with center at the origin.** In this case the inequalities

$$|c_{k_1\dots k_n}| \leq (k_1 c_{10\dots 0}/\Delta^1_{k_1\dots k_n}(D)) + \cdots + (k_{nc_{0\dots 01}}/\Delta^n_{k_1\dots k_n}(D)), \qquad (3)$$

$$k_1 + \cdots + k_n > 1,$$

hold, where

$$\Delta^i_{k_1\dots k_n}(D) = \sup_{(z_1,\dots,z_n)\in D} (|z_1|^{k_1} \cdots |z_{i-1}|^{k_{i-1}} |z_i|^{k_i - 1} |z_{i+1}|^{k_{i+1}} \cdots |z_n|^{k_n})$$

$$(i = 1, \dots, n).$$

In the proof of this proposition an oversight was made, consisting in the passage from the inequality

$$|c_{k_1\dots k_n}| \leq \inf_{(z_1,\dots,z_n)\in D} \left(\frac{k_1 |z_1|\, c_{10\dots 0}}{|z_1|^{k_1} \cdots |z_n|^{k_n}} + \cdots + \frac{k_n |z_n|\, c_{0\dots 01}}{|z_1|^{k_1} \cdots |z_n|^{k_n}} \right),$$

$$k_1 + \cdots + k_n > 1, \qquad (4)$$

to estimate (3) (see ([3], p. 12). Obviously, this passage, generally speaking, does not take place.*** In connection with this, Proposition 1 needs clarification, which is carried out below. Proposition 1, generally speaking, does not hold,**** since for a function satisfying the condition of Proposition 1 (for brevity of notation we take the case of two variables, and as D the unit hypercone $\{|z_1| + |z_2| < 1\}$

$$F(z_1, z_2) = \frac{z_1 + z_2}{(1 - z_1 - z_2)^2}$$

we have

* In ([8a]), in Theorems 3.1, 3.2, in the extremal functions $a_{00}, \bar{a}_{00}$ are printed; it should be $a_{00}M^{-1}, \bar{a}_{00}M^{-1}$.

** For the definition of a typically real function of many variables, see ([3], p. 12).

*** If D is the polycylinder $S\{|z_i| < R_i,\ i = 1, \dots, n\}$, then this passage is valid.

**** Since in the case S the passage from estimate (4) to estimate (3) is valid, in the case S under the condition of Proposition 1 the estimate

$$|c_{k_1\dots k_n}| \leq \frac{k_1 c_{10\dots 0}}{R_1^{k_1 - 1} R_2^{k_2} \cdots R_n^{k_n}} + \cdots + \frac{k_{nc_{0\dots 01}}}{R_1^{k_1} \cdots R_{n-1}^{k_{n-1}} R_n^{k_n - 1}}, \quad k_1 + \cdots + k_n > 1.$$

will be valid.

$|F''_{z_1 z_2}(0,0)| = 4$. But, according to estimate (3), $|F''_{z_1 z_2}(0,0)| \leqslant 2$. At the same time, the following correction of Proposition 1 is possible. If one excludes from consideration the transition from estimate (4) to estimate (3), then the proof of Proposition 1 (see (3), pp. 11, 12) leads to estimate (4). Therefore, under the conditions of Proposition 1, instead of estimate (3) we shall have estimate (4).

4. Consider the domains $\overline{D}_r = r\overline{D}$, $\overline{K}_r = r\overline{K}$, where r is a positive number less than one.

Theorem 4. If the function

$$F(w,z) = \sum_{k=p}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right)^*,$$

$p \geqslant 1$, is regular in the domain D and $|F(w,z)| < 1$ in D, then in $\overline{D}_r$, $|F(w,z)| \leqslant r^p$.

Proof. Let (w_0, z_0) be an arbitrary point of the domain $\overline{D}_r$. Take any number ρ satisfying the condition $r < \rho < 1$, and consider the domain $D_\rho = \rho D$. The point $(w_0, z_0) \in D_\rho$ and, consequently, the point $(w_0\rho^{-1}, z_0\rho^{-1}) \in D$. Hence, in view of the fact that the domain D is complete, it follows that for $|\xi| < 1$ the points $(\xi w_0 \rho^{-1}, \xi z_0 \rho^{-1}) \in D$. Therefore the function

$$F(\xi w_0 \rho^{-1}, \xi z_0 \rho^{-1}) = \sum_{k=p}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} (w_0\rho^{-1})^{k-l} (z_0 \rho^{-1})^l \right) \xi^k$$

as a function of one variable ξ is regular in the disk $|\xi| < 1$, and in it

$$|F(\xi w_0 \rho^{-1}, \xi z_0 \rho^{-1})| < 1.$$

At the same time the expression

$$A_p(w,z) = \sum_{l=0}^{p} a_{p-l,l} w^{p-l} z^l$$

is not identically zero in D_ρ, for, as was noted above, at least one of the coefficients $a_{p-l,l}$ $(l = 0, 1, 2, \ldots, p)$ is different from zero. Consequently, either $A_p(w_0, z_0) \neq 0$, or $A_p(w_0, z_0) = 0$. If $A_p(w_0, z_0) \neq 0$, then, according to the strengthened Schwarz lemma (5),

$$|F(\xi w_0 \rho^{-1}, \xi z_0 \rho^{-1})| \leqslant |\xi|^p \quad (0 < |\xi| < 1).$$

If $A_p(w_0, z_0) = 0$, then, obviously, the strict inequality

$$|F(\xi w_0 \rho^{-1}, \xi z_0 \rho^{-1})| < |\xi|^p \quad (0 < |\xi| < 1)$$

will hold. Consequently, in the end we have

$$|F(\xi w_0 \rho^{-1}, \xi z_0 \rho^{-1})| \leqslant |\xi|^p \quad (0 < |\xi| < 1),$$

whence, in particular, for $\xi = \rho$,

$$|F(w_0, z_0)| \leqslant \rho^p.$$

Passing in the last inequality to the limit as $\rho \to r$, we obtain

$$|F(w_0, z_0)| \leqslant r^p.$$

Finally, since (w_0, z_0) is an arbitrary point of the domain $\overline{D}_r$, the theorem is proved.

In exactly the same way one proves**

Theorem 5. If the function

$$F(w, z) = a_{00} + \sum_{k=p}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right),$$

$p \geqslant 1$, is regular in the domain D and $|F(w, z)| < 1$ in D, then in $\overline{D}_r$

$$|wF'_w(w, z) + zF'_z(w, z)| \leqslant pr^p(1 - r^{2p})^{-1}(1 - |F(w, z)|^2).$$

On the basis of Theorems 4 and 5, one establishes

Theorem 6. Let the function

$$F(w, z) = a_{00} + \sum_{k=p}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right),$$

$p \geqslant 1$, be regular in the domain D. Then, if $\operatorname{Re} F(w, z) > 0$ in D, then in $\overline{D}_r$

$$|wF'_w(w, z) + zF'_z(w, z)| \leqslant 2pr^p(1 - r^{2p})^{-1} \operatorname{Re} F(w, z);$$

if, moreover, a_{00} is real, then in D_r the following estimates also hold:

$$F(0,0)(1 + r^p)^{-1}(1 - r^p) \leqslant \operatorname{Re} F(w, z) \leqslant F(0,0)(1 - r^p)^{-1}(1 + r^p),$$

$$F(0,0)(1 + r^p)^{-1}(1 - r^p) \leqslant |F(w, z)| \leqslant F(0,0)(1 - r^p)^{-1}(1 + r^p),$$

$$|\operatorname{Im} F(w, z)| \leqslant 2F(0,0) r^p (1 - r^{2p})^{-1}.$$

Further, on the basis of Theorem 4 one establishes

Theorem 7. Let the function

$$F(w, z) = \sum_{k=p}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right), \quad p \geqslant 1,$$

* In the presence of such an expansion we shall assume that at least one of the coefficients $a_{p-l,l}$ $(l = 0, 1, 2, \dots, p)$ is different from zero.

** In the proof here the well-known estimate is used ((5), p. 364).

regular in the domain D and $|\operatorname{Re} F(w,z)| < 1$ in D. Then in $\dot{D}_r$ the following estimates hold

$$|\operatorname{Re} F(w,z)| \leqslant 4\pi^{-1} \operatorname{arc\,tg} r^p,$$

$$|F(w,z)| \leqslant 2\pi^{-1} \ln\left[(1+r_p)/(1-r_p)\right],$$

$$|\operatorname{Im} F(w,z)| \leqslant 2\pi^{-1} \ln\left[(1+r^p)/(1-r^p)\right].$$

Remark 3. Theorems 4-7 and their proofs are completely preserved also in the case of the domain K.

5. Let on the interval $0 < t < R$ a single-valued positive function $\varphi(t)$ be continuous and monotonically nonincreasing, with $\lim \varphi(t) = \varphi(0) < \infty$, $\lim_{t\to R-0} \varphi(t) = \varphi(R) = 0$. Consider, in the space of two complex variables (w,z), the domain $D_0 \ni (0,0)$ bounded by the hypersurface $|z| = \varphi(|w|)$, $0 \le |w| \le R$. Obviously, D_0 is a particular case of the domain D. It is easy to see that on the interval $0 < t < R$ the function $I = \Phi(t)$, where $\Phi(t) \equiv t/\varphi(t)$, is continuous and monotonically increasing. Therefore the inverse function $t = \psi(I)$ is single-valued, continuous, and monotonically increasing on the interval $0 < I < \infty$.

On the basis of Theorems 4-7, in the case of the domain D_0 one obtains, respectively, propositions giving at each point $(w,z) \in D_0$ estimates similar in form to the estimates of Theorems 4-7, but with r replaced in them by $\omega(|w|,|z|)$, where $\omega(|w|,|z|)$ is defined by the equality

$$\omega(|w|,|z|) = \begin{cases} |w|R^{-1}, & \text{for } (w,0) \in D_0, \\ |z|/\varphi(0), & \text{for } (0,z) \in D_0, \\ |w|/\psi(|z|^{-1}|w|), & \text{for the remaining points of } D_0^*. \end{cases}$$

In particular, in the case of the domains $C_\alpha\{(a|w|)^{1/\alpha} + (b|z|)^{1/\alpha} < 1\}$ $(a, b,\ a > 0,\ \alpha \leqslant 1)$, $\omega(|w|,|z|) = [(a|w|)^{1/\alpha} + (b|z|)^{1/\alpha}]^\alpha$. Hence[(2)], in the case of the bicylinder $E\{|w| < R_1,\ |z| < R_2\}$,

$$\omega(|w|,|z|) = \begin{cases} |z|R_2^{-1}, & \text{for } (w,z) \in [(|w|R_1^{-1} \leqslant |z|R_2^{-1}) \cap E], \\ |z|R_1^{-1}, & \text{for } (w,z) \in [(|w|R_1^{-1} > |z|R_2^{-1}) \cap E]. \end{cases}$$

All estimates in these propositions, in the case of the domains $\{a|w|+b|z|<1\}$, are sharp[(2)]. In the case of $C_\alpha (0<\alpha<1)$ and E, these estimates are sharp respectively on the sets $(a|w|=b|z|)\cap C_\alpha$ and $(|w|R_1^{-1}=|z|R_2^{-1})\cap E$**, since there exist functions for which they can be attained. Such functions may be obtained from the functions (10), (13), (18), (35) given in (2) by replacing $ae^{i\alpha}w+be^{i\beta}z$, respectively, by the expressions $(ae^{i\alpha}w+be^{i\beta}z)\cdot 2^{\alpha-1}$ and $(e^{i\alpha}wR_1^{-1}+e^{i\beta}zR_2^{-1})\cdot 2^{-1}$.

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* In the case of n variables, a special class of complete circular domains is taken, which was considered by the author in[(8c)].

** The sharpness of these estimates on each set $\{|w|R_1^{-1}=k|z|R_2^{-1}\}\cap E$, where $0<k<1$ and $1<k<\infty$, which was noted in (2), is understood only in the sense that these estimates on each of the indicated sets are attained for the functions given in (2) at the point $(0,0)$. The same also applies to the estimates in (8) (item 4) in the case of E.

Note: Figure translations are in progress. See original paper for figures.