ON SOME INVARIANTS OF THE GROUP OF UNITRIANGULAR MATRICES IN THE SPACE DUAL TO THE SPACE OF ITS LIE ALGEBRA

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Abstract

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ON SOME INVARIANTS OF THE GROUP OF UNITRIANGULAR MATRICES IN THE SPACE DUAL TO THE SPACE OF ITS LIE ALGEBRA

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A. A. Kirillov in the paper ([1]) gave a method for describing irreducible unitary representations of simply connected nilpotent Lie groups, which consists in the following. Let $\mathfrak{G}$ be a simply connected nilpotent Lie group, G its Lie algebra, ρ the adjoint representation of $\mathfrak{G}$ in G. Denote by G' the space dual to G, and by ρ' the representation of $\mathfrak{G}$ in G' dual to ρ. Then the irreducible unitary representations of $\mathfrak{G}$ correspond one-to-one to the orbits in G' with respect to $\rho'(\mathfrak{G})$.

In the present note the problem of classifying the orbits for the group of unitriangular matrices of order n is considered.

1. Let $\mathfrak{G}_n$ be the group of upper triangular matrices of order n with ones on the main diagonal, and G_n the Lie algebra of this group. We identify the space G'_n, dual to G_n, with the space of lower triangular matrices. An element x^{ij} $(i > j)$ of a matrix $x \in G'_n$ will be called an element belonging to the k-th oblique row if $i - j = k$. In what follows we shall denote the elements of the matrix x by $x^{i+k,i}$. The basis $\mathcal{B}_0$ of the space G'_n consists of the coordinate dyads $e_{i+k,i}$ $(i,\ k = 1, 2, \dots, n-1;\ i + k \le n)$, i.e. of matrices with ones at the intersection of the i-th column with the k-th oblique row and zeros in the remaining places. In the space G'_n there is defined a representation $\rho'(\mathfrak{G}_n)$ of the group $\mathfrak{G}_n$ by the formula

$$\rho'(g)x = [gxg^{-1}]_n, \tag{1}$$

where $g \in \mathfrak{G}_n$. In (1) the symbol $[gxg^{-1}]_n$ denotes the lower triangular matrix in which the elements lying below the main diagonal coincide with the corresponding elements of the matrix gxg^{-1}.

Denote $g = \|g_{\alpha\beta}\|$ and $g^{-1} = \|h_{\alpha\beta}\|$. The elements of the basis $\mathcal{B}_0$ are transformed according to the following law:

$$\tilde{e}_{i+k,i} = \rho'(g) e_{i+k,i} = \sum_{\alpha=i+1}^{i+k} \sum_{\beta=i}^{i+k-1} g_{\alpha,i+k}\, h_{i\beta}\, e_{\alpha\beta}. \tag{2}$$

Let $'G_k^i$ be the subspace in G_n' consisting of matrices $x = \|x^{\alpha\beta}\|$ for which $x^{\alpha\beta} = 0$ for $\alpha > i + k$ and $\beta < i$. It follows from (2) that $'G_k^i$ is invariant with respect to $\rho'(\mathfrak{G}_n)$ and contains an orbit Ω_k^i which is a transitive submanifold in it. In what follows we shall call the subspace $'G_k^i$ the $\binom{i}{k}$-th block. The coordinates $\tilde{x}^{\alpha\beta}$ of the image $\tilde{x}$ of an element $x \in {}'G_k^i$ under transformations from $\rho'(\mathfrak{G}_n)$ are found by the formula

$$\tilde{x}^{\alpha\beta} = \sum_{\gamma=\alpha}^{i+k} \sum_{\delta=i}^{\beta} g_{\alpha\gamma} x^{\gamma\delta} h_{\delta\beta}. \tag{2a}$$

It is not hard to show that the orbit Ω_k^i, contained in $'G_k^i$, is for it an orbit of general position (see (1), p. 106). We arrive at the following result:

a) By specifying the element $e_{i+k,i}$ of the basis $\mathcal{B}_0$ of the space G_n', one singles out in the latter the subspace $'G_k^i$, invariant with respect to $\rho'(\mathfrak{G}_n)$ and containing the $\binom{i}{k}$-th class of orbits. The orbit Ω_k^i from the $\binom{i}{k}$-th class is an algebraic surface defined by the equations:

$$_\nu\Delta_k^i = \text{const} \neq 0 \quad (\nu = 1, 2, \dots, [(k+1)/2]), \tag{3}$$

where $_\nu\Delta_k^i$ is the determinant of the minor of order ν of the left lower corner of the $\binom{i}{k}$-th block.

$$\dim \Omega_k^i = k(k+1)/2 - [(k+1)/2].$$

In what follows, the element $e_{i+k,i}$ of the basis $\mathcal{B}_0$, defining the $\binom{i}{k}$-th class of orbits, will be called the leading vector of the class of orbits.

2. Let m elements of the basis $\mathcal{B}_0$ be taken,

$$e_{i_1+k_1,i_1}, \quad e_{i_2+k_2,i_2}, \dots; \quad e_{i_m+k_m,i_m}, \tag{4}$$

where the numbers i_ε and k_ε satisfy the conditions:

$$1 \leq i_1 < i_2 < \dots < i_m \leq n-1,$$

$$i_\varepsilon + k_\varepsilon < i_{\varepsilon+1} + k_{\varepsilon+1} \quad (\varepsilon = 1, 2, \dots, m-1), \quad i_m + k_m \le n. \tag{5}$$

From the transformation law for coordinate vectors it follows that m leading vectors (4) single out in G'_n a subspace

$$'G^{i_1 i_2 \dots i_m}_{k_1 k_2 \dots k_m} = \bigcap_{\alpha=1}^{m} {}'G^{i_\alpha}_{k_\alpha},$$

invariant with respect to $\rho'(\mathfrak{G}_n)$ and containing the

$$\begin{pmatrix} i_1\, i_2\, \dots\, i_m \\ k_1\, k_2\, \dots\, k_m \end{pmatrix}$$

-th class of orbits. The orbit

$$\Omega^{i_1 i_2 \dots i_m}_{k_1 k_2 \dots k_m},$$

as a transitive manifold in

$$'G^{i_1 i_2 \dots i_m}_{k_1 k_2 \dots k_m},$$

is singled out in it by a system of functions of the coordinates $x^{\alpha\beta}$ of an element

$$x \in {}'G^{i_1 i_2 \dots i_m}_{k_1 k_2 \dots k_m},$$

invariant with respect to $\rho'(\mathfrak{G}_n)$. Here two systems of such invariant functions are indicated for $m = 2$.

We first indicate two special cases, interesting in that these are the only possibilities when the orbits are linear subspaces in G'_n. For $k_1 = k_2 = \dots = k_m = 1$ we obtain the class of zero-dimensional orbits (the orbit is a point in G'_n).

For the case when the leading vectors (4) satisfy the conditions

$$k_1 = k_2 = \dots = k_m = 2, \quad i_{\varepsilon+1} = i_\varepsilon + 1 \quad (\varepsilon = 1, 2, \dots, m-1), \tag{6}$$

the following assertion is true:

β) m leading vectors (4) under conditions (6) determine the

$$\begin{pmatrix} i_1\, i_2\, \dots\, i_m \\ k_1\, k_2\, \dots\, k_m \end{pmatrix}$$

-th class of orbits. The orbit

$$\Omega^{i_1 i_2 \dots i_m}_{k_1 k_2 \dots k_m}$$

is:

a) for even m, an m-dimensional plane defined by the equations

$$x^{\alpha+2,\alpha} = \text{const} \neq 0 \quad (\alpha = 1, 2, \dots, m),$$

$$A_1 x^{i_1+1,i_1} + A_2 x^{i_3+1,i_3} + \dots + A_{m/2+1} x^{m+2,i_m+1} = B, \tag{7}$$

where $A_1, A_2, \dots, A_{m/2+1}, B$ are constants determined by the choice of the vector

$$x \in {}'G^{i_1 i_2 \dots i_m}_{k_1 k_2 \dots k_m}.$$

b) for odd m, an $(m+1)$-dimensional plane defined by the equations

$$x^{\alpha+2,\alpha} = \text{const} \neq 0 \qquad (\alpha = 1, 2, \dots, m).$$

3. Let us return again to the case $m = 1$. From the elements of the matrix $x \in {}'G^i_k$ we find two series of determinants. To this end, in the matrix $x \in {}'G^i_k$ we single out the $(i+p)$-th row and, from the elements that are not identically zero, form determinants ${}^{\alpha}_{p}D^i_k$ of order p ($p = 2, 3, \dots, [(k+1)/2]$) ($\alpha = i+p, i+p+1, \dots, i+k-p+1$) in the following way: the first row of the determinant consists of the first p elements of the α-th row of the matrix x, while the remaining $p-1$ rows are the last $p-1$ rows of the matrix x, taken in the same order as in the matrix x, and containing the first p elements. Thus the determinants ${}^{\alpha}_{p}D^i_k$ have the form

$$ {}^{\alpha}_{p}D^i_k = \begin{vmatrix} x^{\alpha,i} & x^{\alpha,i+1} & \dots & x^{\alpha,i+p} \\ x^{i+k-(p-1),i} & x^{i+k-(p-1),i+1} & \dots & x^{i+k-(p-1),i+p} \\ x^{i+k-(p-2),i} & x^{i+k-(p-2),i+1} & \dots & x^{i+k-(p-2),i+p} \\ \dots & \dots & \dots & \dots \\ x^{i+k,i} & x^{i+k,i+1} & \dots & x^{i+k,i+p} \end{vmatrix} \tag{8}$$

Similarly, we define the second series of determinants of order p, ${}^{\alpha}_{p}Q^i_k$ ($\alpha = i+p-1, i+p, \dots, i+k-p$). To do this, in the matrix $x \in {}'G^i_k$ we single out the last p rows. Then, in the determinant ${}^{\alpha}_{p}Q^i_k$, the $p-1$ columns are the first $p-1$ columns of the selected p rows, and the p-th column is the α-th column in the selected p rows. Thus

$$
{}^{\alpha}_{p}Q^{i}_{k} = \begin{vmatrix} x^{i+k-p,i} & x^{i+k-p,i+1} & \dots & x^{i+k-p,i+p-1} & x^{i+k-p,\alpha} \\ x^{i+k-p+1,i} & x^{i+k-p+1,i+1} & \dots & x^{i+k-p+1,i+p-1} & x^{i+k-p+1,\alpha} \\ \dots & \dots & \dots & \dots & \dots \\ x^{i+k,i} & x^{i+k,i+1} & \dots & x^{i+k,i+p-1} & x^{i+k,\alpha} \end{vmatrix}. \quad (9)
$$

It is easy to see that for $p = [(k+1)/2]$

$$
{}^{\alpha}_{p}D^{i}_{k} = {}^{\alpha}_{p}Q^{i}_{k} = {}_{[(k+1)/2]}\Delta^{i}_{k}.
$$

Moreover, by definition, we put

$$
{}^{\alpha}_{1}D^{i}_{k} = x^{\alpha i}, \qquad {}^{\alpha}_{1}Q^{i}_{k} = x^{i+k,\alpha}.
$$

For the determinants ${}^{\alpha}_{p}D^{i}_{k}$ and ${}^{\alpha}_{p}Q^{i}_{k}$, denote by ${}^{\alpha}_{p}\widetilde{D}^{i}_{k}$ and ${}^{\alpha}_{p}\widetilde{Q}^{i}_{k}$ their images under the transformations from $\rho'(\mathfrak{G}_n)$. Then

$$
{}^{\alpha}_{p}\widetilde{D}^{i}_{k} = \sum_{\beta=\alpha}^{i+k-p-1} g_{\alpha\beta}\, {}^{\beta}_{p}D^{i}_{k}, \quad (10)
$$

$$
{}^{\alpha}_{p}\widetilde{Q}^{i}_{k} = \sum_{\beta=i+p-1}^{\alpha} {}^{\beta}_{p}Q^{i}_{k}\, h_{\beta\alpha}. \quad (11)
$$

Now one can describe the first type of functions of the coordinates $x^{\alpha\beta}$ of an element $x \in {}'G^{i_1 i_2}_{k_1 k_2}$, invariant with respect to $\rho'(\mathfrak{G}_n)$. The subspace ${}'G^{i_1 i_2}_{k_1 k_2}$ consists of matrices $x = \|x^{\alpha\beta}\|$ in which $x^{\alpha\beta} = 0$ for all α when $i_1 > \beta$, for $i_2 + k_2 < \alpha < i_1 + k_1$ when $i_1 \le \beta < i_2$, and for all β when $\alpha > i_2 + k_2$, while for the remaining values of α and β the $x^{\alpha\beta}$ are arbitrary.

The subspace ${}'G^{i_1 i_2}_{k_1 k_2}$ can be represented as the sum of two subspaces

$$
{}'G^{i_1 i_2}_{k_1 k_2} = {}'G^{i_1}_{k_1} + {}'G^{i_2}_{k_2}. \quad (12)
$$

If ${}'G^{i_1}_{k_1} \cap {}'G^{i_2}_{k_2} = 0$, then the subspace ${}'G^{i_1 i_2}_{k_1 k_2}$ decomposes into the direct sum of the subspaces ${}'G^{i_1}_{k_1}$ and ${}'G^{i_2}_{k_2}$, which leads to the decomposition of the orbit $\Omega^{i_1 i_2}_{k_1 k_2}$ into the orbits $\Omega^{i_1}_{k_1}$ and $\Omega^{i_2}_{k_2}$.

Let ${}'G^{i_1}_{k_1} \cap {}'G^{i_2}_{k_2} \neq 0$. In the matrix $x \in {}'G^{i_1 i_2}_{k_1 k_2}$ we single out $i_2 - i_1$ first columns, beginning with the i_1-st, and $i_2 + k_2 - i_1 - k_1$ last rows, beginning with the $(i_1 + k_1 + 1)$-st, and define two systems of determinants of orders p and r, respectively:

$${}^{\alpha}_{p}D^{i_1}_{k_1} \quad (\alpha = i_1{+}p,\ i_1{+}p{+}1, \dots,\ i_1{+}k_1{-}p{+}1), \qquad {}^{\alpha}_{r}Q^{i_2}_{k_2} \quad (\alpha = i_2{+}r{-}1,\ i_2{+}r, \dots,\ i_2{+}k_2{-}r), \tag{13}$$

where $p = 1, 2, \dots, i_2 - i_1$ if $i_2 - i_1 < [(k_1+1)/2]$, and $p = 1, 2, \dots, [(k_1+1)/2]$ if $i_2 - i_1 \geq [(k+1)/2]$; $r = 1, 2, \dots, i_2 + k_2 - i_1 - k_1$ if $i_2 + k_2 - i_1 - k_1 < [(k_2+1)/2]$, and $r = 1, 2, \dots, [(k_2+1)/2]$ if $i_2 + k_2 - i_1 - k_1 \geq [(k_2+1)/2]$.

The following assertion holds:

γ) The expression

$$\sum_{\alpha=i_2+r-1}^{i_1+k_1-p+1} {}^{\alpha}_{r}Q^{i_2}_{k_2} \cdot {}^{\alpha}_{p}D^{i_1}_{k_1}, \tag{14}$$

formed from the determinants ${}^{\alpha}_{r}D^{i_1}_{k_1}$ and those determinants ${}^{\alpha}_{r}Q^{i_2}_{k_2}$ whose order r, for the given p, satisfies the condition $i_2 + r - 1 \leq i_1 + k_1 - p + 1$, remains invariant under all transformations from $\rho'(\mathfrak{G}_n)$.

4. For the same case $m = 2$ we carry out the following constructions: we complete the subspace $'G^{i_1 i_2}_{k_1 k_2}$ to $'G^{i_1}_{i_2+k_2-i_1}$, replacing the elements $x^{\alpha\beta}$ $(i_1 + k_1 < \alpha \leq i_2 + k_2;\ i_1 \leq \beta < i_2)$, identically equal to zero, by elements $\bar{x}^{\alpha\beta}$ with the same values α and β from the square of the matrix x, and leaving the remaining ones unchanged. In the subspace $'G^{i_1}_{i_2+k_2-i_1}$ the same subgroup $'\rho^{i_1 i_2}_{k_1 k_2}(\mathfrak{G}_n)$ of the group $\rho'(\mathfrak{G}_n)$ acts as in the subspace $'G^{i_1 i_2}_{k_1 k_2}$. The elements $\bar{x}^{\alpha\beta}$ $(i_1 + k_1 < \alpha \leq i_2 + k_2;\ i_1 \leq \beta < i_2)$ under the action of transformations from $'\rho^{i_1 i_2}_{k_1 k_2}(\mathfrak{G}_n)$ transform according to the formula

$$\bar{x}^{\alpha\beta} = \sum_{\gamma=\alpha}^{i_2+k_2} \sum_{\delta=i_1}^{\beta} g_{\alpha\gamma} \bar{x}^{\gamma\delta} h_{\delta\beta}. \tag{15}$$

From formula (15) it follows directly:

δ) In the subspace $'G^{i_1 i_2}_{k_1 k_2}$ the functions

$${}_{\nu}\Delta^{i_1}_{i_2+k_2-i_1}, \tag{16}$$

where ${}_{\nu}\Delta^{i_1}_{i_2+k_2-i_1}$ is the determinant of the minor of order ν of the lower left corner of the block $\begin{pmatrix} i_1 \\ i_2 + k_2 - i_1 \end{pmatrix}$, are invariant with respect to $\rho'(\mathfrak{G}_n)$, and

$$\nu = \begin{cases} 1, 2, \dots, i_2 + k_2 - i_1 - k_1, & \text{if } k_1 < k_2, \\ 1, 2, \dots, i_2 - i_1, & \text{if } k_1 \geq k_2, \end{cases}$$

and always $\nu \leq [(i_2 + k_2 - i_1 + 1)/2]$.

It is easy to see that in $'G^{i_1 i_2}_{k_1 k_2}$, besides invariants of types (14) and (16), there are invariants of type (3), defined for $'G^{i_1}_{k_1} \subset {'G^{i_1 i_2}_{k_1 k_2}}$ and $G^{i_2}_{k_2} \subset {'G^{i_1 i_2}_{k_1 k_2}}$, with the known restrictions on ν.

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1. A. A. Kirillov, UMN, 17, no. 4 (1962).

Note: Figure translations are in progress. See original paper for figures.