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Abstract

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THEORY OF ELASTICITY

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BENDING BY A TRANSVERSE FORCE OF COMPOSITE ANISOTROPIC CYLINDRICAL BODIES WITH A SLIGHTLY CURVED AXIS

(Presented by Academician N. I. Muskhelishvili on 9 X 1964)

The problem of bending by a transverse force of cylindrical bodies with a slightly curved axis in the case of an isotropic material has been studied in papers (1–3).

1. Consider a rectangular Cartesian coordinate system $Ox_1x_2x_3$ and an anisotropic cylindrical body with a curved axis t_0', bounded by the planes $x_3 = 0$, $x_3 = l$ $(l > 0)$ and by the surface

$$f_{m+1}(x_1 + {}^1/_2\varepsilon x_3^2,\ x_2) = 0. \qquad (1,1)$$

Let the body t_0' have cavities whose surface equations are of the form

$$f_j(x_1 + {}^1/_2\varepsilon x_3^2,\ x_2) = 0 \quad (j = 1, 2, \dots, m), \qquad (1,2)$$

where ε is a small parameter, whose squares and higher powers may be neglected.

Assume that the cavities are filled with other anisotropic materials. The bodies formed by them shall be denoted by t_j' $(j = 1, 2, \dots, m)$. Let the bodies t_j' $(j = 1, 2, \dots, m)$ be bonded to the body t_0' along the surfaces $(1, 2)$. The resulting body t' will be called a composite anisotropic cylindrical body with a slightly curved axis.

Let each of the anisotropic materials composing the body t' have one plane of elastic symmetry perpendicular to the axis Ox_3. Then the generalized Hooke' s law can be written in the form

$$e_{ii} = E^{-1}(\sigma_{1i}\tau_{11} + \sigma_{2i}\tau_{22} + \sigma_{3i}\tau_{12} - \sigma_i\tau_{33}),$$

$$e_{12} = E^{-1}(\sigma_{13}\tau_{11} + \sigma_{23}\tau_{22} + \sigma_{33}\tau_{12} - \sigma_3\tau_{33}), \tag{1,3}$$

$$e_{33} = E^{-1}(\tau_{33} - \sigma_1\tau_{11} - \sigma_2\tau_{22} - \sigma_3\tau_{12}), \qquad e_{i3} = (-1)^i A_*^{-1}(A_{k5}\tau_{23} - A_{4k}\tau_{13}),$$

$$A_* = (A_{44}A_{55} - A_{45}) > 0 \quad (i = 1, 2;\ k = 3 + i)$$

or

$$\tau_{ii} = A_{1i}e_{11} + A_{2i}e_{22} + A_{3i}e_{33} + A_{6i}e_{12},$$

$$\tau_{12} = A_{16}e_{11} + A_{26}e_{22} + A_{36}e_{33} + A_{66}e_{12}, \tag{1,4}$$

$$\tau_{13} = A_{55}e_{13} + A_{45}e_{23}, \qquad \tau_{23} = A_{45}e_{13} + A_{44}e_{23} \quad (i = 1, 2, 3),$$

where τ_{ij} and e_{ij} are the components of stress and strain; E, σ_{ij}, and A_{ij} are elastic constants.

In the problem under consideration, the stress components τ_{ij} and strains e_{ij} must satisfy the equilibrium equations, the Saint-Venant compatibility conditions, and the boundary conditions

$$\left[\sum_{k=1}^{3} \tau_{ik}\cos(\nu, x_k)\right]_j - [\text{ident}]_0 = 0 \quad (j = 1, 2, \dots, m + 1;\ [\ ]_{m+1} = 0) \tag{1,5}$$

on the surfaces $(1,1)$ and $(1,2)$, $(1,5)$,

$$[u_i]_j = [u_i]_0 \quad (j = 1, 2, \dots, m;\ i = 1, 2, 3) \quad \text{on the surfaces } (1,2),$$

where ν is the normal to the surfaces $(1,1)$ and $(1,2)$, u_i are the displacement components, and the symbols $[\ ]_j$ and $[\ ]_0$ denote the limiting values on the indicated surfaces of the expressions enclosed in brackets, taken respectively from the regions t'_j $(j = 1, 2, \dots, m)$ and t'_0. The end conditions will be discussed below.

Make the change of variables (^{1})

$$\xi_1 = x_1 + {}^1\!/_2\varepsilon x_3^2, \qquad \xi_2 = x_2, \qquad \xi_3 = x_3. \tag{1,6}$$

Then equations (1.1) and (1.2), respectively, take the form

$$f_{m+1}(\xi_1, \xi_2) = 0, \tag{1.7}$$

$$f_j(\xi_1, \xi_2) = 0 \qquad (j = 1, 2, \dots, m), \tag{1.8}$$

i.e., in the system $O\xi_1\xi_2\xi_3$ the body t' will correspond to a composite anisotropic cylindrical body t.

To within terms of order ε^2 we may write [1]

$$\cos(\nu, x_i) = \cos(n, \xi_i), \qquad \cos(\nu, x_3) = \varepsilon\xi_3 \cos(n, \xi_1); \tag{1.9}$$

$$\partial/\partial x_i = \partial_i, \qquad \partial/\partial x_3 = \partial_3 + \varepsilon\xi_3\partial_1 \quad (i = 1, 2;\ \partial_j = \partial/\partial\xi_j),$$

where n is the normal to the surfaces (1.7) and (1.8).

It is assumed that the origin of the coordinate system $O\xi_1\xi_2\xi_3$ is placed at the generalized center of inertia of the "lower" base (i.e., for $\xi_3 = 0$) of the composite anisotropic cylindrical body t, while the axes $O\xi_1$ and $O\xi_2$ are directed along the generalized principal axes of inertia of this base [4, 5].

Denote by $\tau_{ij}^{(2)}$ and $u_i^{(2)}$ the stress and displacement components of the plane auxiliary problem for the composite anisotropic cylindrical body t [4], in which the external forces on the surface (1.8) are zero, the stress vector in passing through the surface (1.9) remains continuous, and the displacement components undergo discontinuities of the form

$$[u_1^{(2)}]_j - [u_1^{(2)}]_0 = -\frac{1}{2}\left[(\sigma_1^{(j)} - \sigma_1^{(0)})\xi_1^2 - (\sigma_2^{(j)} - \sigma_2^{(0)})\xi_2^2\right],$$

$$[u_2^{(2)}]_j - [u_2^{(2)}]_0 = -(\sigma_2^{(j)} - \sigma_2^{(0)})\xi_1\xi_2 - \frac{1}{2}(\sigma_3^{(j)} - \sigma_3^{(0)})\xi_1^2,$$

$$(j = 1, 2, \dots, m), \tag{1.10}$$

where $\sigma_i^{(j)}$ are the values of the elastic constants in the region t_j $(j = 0, 1, \dots, m)$.

Introduce the notation:

$$E_* = E - A_{55}\sigma_1 - A_{44}\sigma_2 - A_{45}\sigma_3, \qquad \beta_{j0} = A_{55}\beta_{j1} + A_{44}\beta_{j2} + A_{45}\beta_{j3} + \sigma_3,$$

$$\beta_{ij} = E^{-1}(\sigma_{ij} - \sigma_i\sigma_j) \qquad (i, j = 1, 2, 3),$$

$$\tau_{33}^{(2)} = \sigma_1\tau_{11}^{(2)} + \sigma_2\tau_{22}^{(2)} + \sigma_3\tau_{12}^{(12)}, \qquad \sigma_i^* = A_{\gamma 5}\sigma_1 + A_{4\gamma}\sigma_3,$$

$$\theta_i = A_{\gamma 5}u_1^{(2)} + A_{4\gamma}u_2^{(2)}, \qquad d_i = A_{\gamma 5}\partial_1 + A_{4\gamma}\partial_2 \quad (i = 1, 2;\ \gamma = 6 - i); \tag{1.11}$$

$$\theta = (A_{55} + A_{13})\partial_1 u_1^{(2)} + (A_{44} + A_{23})\partial_2 u_2^{(2)} + (A_{45} + A_{36})(\partial_1 u_2^{(2)} + \partial_2 u_1^{(2)}),$$

$$\Delta_1 = A_{44}\partial_2^2 + 2A_{45}\partial_1\partial_2 + A_{55}\partial_1^2,$$

$$\Delta_j^* = \beta_{1j}\partial_2^2 - \beta_{3j}\partial_1\partial_2 + \beta_{2j}\partial_1^2 \qquad (j = 0, 1, 2),$$

$$\Delta_2 = \beta_{22}\partial_1^4 + (2\beta_{12} + \beta_{33})\partial_1^2\partial_2^2 - 2\beta_{13}\partial_1\partial_2^3 - 2\beta_{23}\partial_1^3\partial_2 + \beta_{11}\partial_2^4,$$

$$\tau_{in} = \sum_{j=1}^{2} \tau_{ij}\cos(n, \xi_j) \qquad (i = 1, 2, 3),$$

where E, σ_{ij}, σ_i, and A_{kl} are elastic constants.

2. Let the external forces applied to the base $x_3 = l$ of the composite anisotropic cylindrical body t' with a slightly curved axis be equivalent to a bending force W directed along the axis $O\xi_1$. Taking into account the relations (1.6), we shall seek the solution of the problem in the form

$$u_i^0 = a\varepsilon u_i + \frac{a}{2}(i-2)\left[(i-1)\chi - \left(\sqrt{2^{\,i-1}}\tau\xi_3 - \frac{\xi_3^2}{k}\right)\xi_k\right] +$$
$$+(-1)^i\alpha\tau 2^{\,i-2}\xi_3\xi_\alpha +$$
$$+\alpha 2^{\,i-2}a(l-\xi_3)\left[u_i^{(2)} + \frac{1}{2}\left(\frac{2}{a}\sigma_i\xi_1\xi_i + (-1)^i\sigma_{i+1}\xi_\alpha^2\right)\right], \qquad (2.1)$$

where

$$i = 1, 2, 3; \qquad k = 4 - i; \qquad \alpha = 3 - i; \qquad \chi = \frac{\tau}{a}\varphi - \psi,$$

$$a = W\left[\sum_{j=0}^{m}\iint_{\Omega_j}\left(E\xi_1^2 - \tau_{33}^{(2)}\xi_1\right)d\Omega_j\right]^{-1}, \qquad (2.2)$$

ψ is the bending function, φ is the torsion function, and τ is the degree of twisting of the composite cylindrical body t [4]; ε is the above-mentioned small

parameter; Ω_j is the region of the lower base of the cylindrical body t; u_i are the components of the "additional displacement" to be determined; $\tau_{ij}^{(2)}$ and $u_i^{(2)}$ are the solution of the plane auxiliary problem (see (1.1) and (1.2)). As was said, we shall neglect terms of order ε^2 and higher.

The stress components τ_{ij}^0 corresponding to the displacements (2.1) will have the form:

$$\tau_{ii}^0 = a\left[(l-\xi_3)\tau_{ii}^{(2)} + \varepsilon A_{i3}\xi_3\partial_1\chi + \varepsilon\tau_{ii} - \frac{1}{2}E(i-1)(i-2)(l-\xi_3)\xi_1\right],$$

$$(i = 1, 2, 3), \qquad \tau_{12}^0 = a\left[(l-\xi_3)\tau_{12}^{(2)} + \varepsilon A_{36}\xi_3\partial_1\chi + \varepsilon\tau_{12}\right],$$

$$\tau_{i3}^0 = a\left[d_i\chi - \frac{1}{2}A_{5\gamma}(\gamma_1\xi_1^2 - \sigma_2\xi_2^2) - A_{4\gamma}\left(\sigma_2\xi_1\xi_2 + \frac{1}{2}\sigma_3\xi_1^2\right) - \theta_i\right] -$$

$$-\tau(A_{5\gamma}\xi_2 - A_{4\gamma}\xi_1) + \varepsilon A_{4\gamma}\tau\xi_3^2 + a\varepsilon\tau_{i3} + \qquad (2.3)$$

$$+a\varepsilon(l-\xi_3)\xi_3\left[A_{5\gamma}\sigma_1\xi_1+A_{4\gamma}(\sigma_2\xi_2+\sigma_3\xi_1)+\partial_1\theta_i\right]\qquad(i=1,2;\ \gamma=6-i),$$

where τ_{ij} are the stress components corresponding to the unknown displacements u_i, while d_i and θ_i are defined by the equalities (1.11).

Substituting the quantities (2.3) and the corresponding strain components e_{ij} into the equilibrium equations and the Saint-Venant compatibility conditions, we obtain that e^0_{ij}, with respect to the variables ξ_i, must satisfy the Saint-Venant compatibility conditions, while τ_{ij} in each of the regions t_j $(j=0,1,\dots,m)$ must satisfy the following equations:

$$\sum_{j=1}^{3}\partial_j\tau_{ij}+\xi_3\left[(A_{i3}+A_{\gamma\gamma})\partial_i\partial_1\chi+(A_{36}+A_{45})\partial_\alpha\partial_1\chi-3A_{\gamma5}(\sigma_1\xi_1+\partial_1u_1^{(2)})+\right.$$

$$\left.+3A_{4\gamma}\left(\frac{\tau}{a}-\sigma_2\xi_2-\sigma_3\xi_1-\partial_1u_2^{(2)}\right)\right]+l\left[A_{4\gamma}(\sigma_2\xi_2+\sigma_3\xi_1+\partial_1u_2^{(2)})+\right.$$

$$\left.+A_{\gamma5}(\sigma_1\xi_1+\partial_1u_1^{(2)})\right]=0,\tag{2.4}$$

$$\sum_{j=1}^{3}\partial_j\tau_{3j}+A_{33}\partial_1\chi+\xi_3(l-\xi_3)(E_*+\partial_1\theta)=0\qquad(i=1,2;\ \gamma=6-i;\ \alpha=3-i),$$

where E_* and θ are defined by the equalities (1.11). Using the results of (5,6), as a particular solution of equations (2.4) we may take the quantities

$$\tau^1_{ii}=\sum_{j=0}^{1}\xi_3^j\left(\partial_\alpha^2\Psi_j-2^jA_{\gamma\gamma}\Phi_{j+1}\right)+(3\xi_3-l)\left(\frac{1}{a}\sigma_i^*\xi_1\xi_i+A_{4\gamma}\sigma_2\xi_2\xi_i+\right.$$

$$\left.+\partial_1\int\theta_1\,d\xi_i\right)-\left[3A_{4\gamma}\frac{\tau}{a}\xi_i+(A_{i3}+A_{\gamma\gamma})\partial_1\chi\right]\xi_3,$$

$$\tau^1_{12}=-\sum_{j=0}^{1}\xi_3^j\left(\partial_1\partial_2\Psi_j+2^jA_{45}\Phi_{j+1}\right)-\xi_3(A_{36}+A_{45})\partial_1\chi,$$

$$\tau^1_{33}=\sum_{j=1}^{2}\left(\sigma_1\tau^1_{jj}+jE\xi_3^{j-1}\Phi_i\right)+\sigma_3\tau^1_{12},\qquad\tau^1_{i3}=\sum_{j=0}^{2}\left(\xi_3d_i\Phi_j-\right.$$

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$$-j\frac{3}{2^j}\sigma_j\partial_\alpha \iint \theta_j \, d\xi_2 \, d\xi_i\Big)-\frac{1}{4}\Big[3^{i-1}\xi_1\xi_i(\sigma_1\sigma_1^*\xi_1+3^{2-i}\sigma_2\sigma_2^*\xi_2)-3\frac{\tau}{a}\,(2^{2-i}A_{44}\sigma_2\xi_2+$$

$$+2^{i-1}A_{45}\sigma_1\xi_1)\,\xi_i+3^{2-i}\sigma_2\xi_1\xi_i\,(A_{44}\sigma_2\xi_2+3^{i-1}A_{45}\sigma_1\xi_1)]-$$

$$-\frac{1}{2}\Big[E_*\int \Phi_2\, d\xi_i+(-1)^\alpha(E_*-A_{33})\partial_1\int \chi\, d\xi_i+\sigma_i\int \partial_\alpha\Psi_1\, d\xi_i+$$

$$+\sigma_\alpha\partial_2\Psi_1-\sigma_3\partial_\alpha\Psi_1]+(-1)^\alpha\partial_\alpha\Phi_3 \qquad (\alpha=3-i;\ \gamma=6-i;\ i=1,2), \quad (2.5)$$

where E_*, σ_i^*, θ_i, and d_i are defined by the equalities (1.11), while the functions $\Phi_i(\xi_1,\xi_2)$ and $\Psi_i(\xi_1,\xi_2)$ are particular solutions, respectively, of the equations

$$\Delta_1\Phi_i=X_i(\xi_1,\xi_2)\quad (i=0,1,2,3); \qquad \Delta_2\Psi_j=Y_j(\xi_1,\xi_2)\quad (i=0,1),$$

where

$$X_0=-A_{33}\partial_1\chi, \qquad X_1=-l(E_*+\partial_1\theta), \qquad X_2=E_*+\partial_1\theta,$$

$$X_3=\left\{\partial_3\left[\partial_2\int\left(\frac{A_{44}}{2}\tau_{33}^1-A_*e_{11}^1\right)d\xi_1-\partial_1\int\left(\frac{A_{55}}{2}\tau_{33}^1+A_*e_{22}^1\right)d\xi_2\right]\right\}_{\xi_3=0},$$

$$Y_0=\Delta_0^*\Phi_1+l\Delta_1^*\theta_1+l\Delta_2^*\int\partial_1\theta_2\, d\xi_2+l(\beta_{12}\sigma_1^*-\beta_{23}\sigma_2^*-\beta_{13}A_{45}\sigma_2),$$

$$Y_1=3\Delta_1^*\theta_1-3\Delta_2^*\int\partial_1\theta_2\, d\xi_2+\Delta_0^*\partial_1\chi+2\Delta_0^*\Phi_2-3\beta_{12}(\sigma_1^*+A_{44}\sigma_2)+$$

$$+3\beta_{13}A_{45}\sigma_2+3\beta_{23}\sigma_2^*,$$

where E_*, σ_i^*, Δ_i, Δ_j^*, and τ_{ij}^1 are defined by the equalities (1,11) and (2,5); e_{ij}^1 are the strain components related to τ_{ij}^1 by the equalities (1,3).

We shall seek the unknown stress components τ_{ij} entering expressions (2,3) in the form of a sum:

$$\tau_{ij}=\tau_{ij}^1+\tau_{ij}^* \qquad (i,j=1,2,3), \qquad (2,6)$$

where τ^1_{ij} are defined by the equalities (2,5), while τ^*_{ij} are to be determined.

Substituting expressions (2,3) and the corresponding strain components e^0_{ij} into the equilibrium equations, the Saint-Venant compatibility conditions, and the boundary conditions (1,5), and taking (2,6) into account, we obtain that in each of the regions t_j $(j = 0, 1, \dots, m)$ the stress components τ^*_{ij} and strains e^*_{ij} must satisfy, with respect to the variables ξ_i, the equilibrium equations and the Saint-Venant compatibility conditions, as well as the following boundary conditions:

$$[\tau^*_{in} + \tau^1_{in} + \tau^*_i]_j - [\text{ident}]_0 = 0 \quad \text{on the surfaces (1,7) and (1,8),}$$

$$(i, j = 1, 2, \dots, m+1;\ [\ ]_{m+1} = 0); \tag{2,7}$$

$$[u^*_i + u^1_i]_j - [\text{ident}]_0 = 0 \quad \text{on the surfaces (1,8)}$$

$$(j = 1, 2, \dots, m;\ i = 1, 2, \dots),$$

where τ_{in} are defined by the equalities (1,11); u^1_i are the displacement components corresponding to the particular solution τ^1_{ij}, and τ^*_i are defined by the equalities

$$\tau^*_i = \left[\frac{\tau}{a}(A_{5\gamma}\xi_2 - A_{4\gamma}\xi_1) - \xi_3\left(d_i\chi - \frac{1}{2}\sigma^*_i\xi_1^2 - A_{4\gamma}\xi_1\xi_2 + \right.\right.$$

$$\left.\left. + \frac{1}{2}A_{5\gamma}\sigma_2\xi_2^2 - \theta_i\right)\right] \cos(n, \xi_1) - \xi_3(A_{i3} + A_{36})\partial_1\chi \cos(n, \xi_i)$$

$$(i = 1, 2;\ \gamma = 6 - i),$$

$$\tau^*_3 = -\sum_{i=1}^{2}\left[\xi_3(l - \xi_3)(\sigma^*_i\xi_1 + A_{4\gamma}\sigma_2\xi_2 + \partial_1\theta_i) + \right.$$

$$\left. + \frac{\tau}{a}A_{4\gamma}\xi_3^2\right] \cos(n, \xi_i) + E(l - \xi_3)\xi_1 \cos(n, \xi_1).$$

Thus, τ^*_{ij} is the solution of Almanzi' s problem, for the composite anisotropic cylindrical body t, given in work (5).

To satisfy the end conditions, as in the case of an isotropic body (1), it is sufficient to superpose on expressions (2,6) the corresponding solution of the Saint-Venant problem for the composite anisotropic cylindrical body t (4).

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