ON THE RECOVERY OF THE CHARACTERISTICS OF A SYSTEM FROM OBSERVATIONS OF THE OUTPUT FLOW

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Abstract

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MATHEMATICS

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ON THE RECOVERY OF THE CHARACTERISTICS OF A SYSTEM FROM OBSERVATIONS OF THE OUTPUT FLOW

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1. In practical problems connected with evaluating the efficiency of complex systems, one often has to face the need to reveal the structure and estimate the parameters of a system from observations of the outgoing flow of demands. The necessity of solving such problems was pointed out by B. V. Gnedenko, D. Kendall, and others. In particular, D. Kendall, in his report at the Seventh All-Union Conference on Probability Theory and Mathematical Statistics, mentioned the problem of estimating the distribution of the service-time duration of demands from the outgoing flow in the case of a single-server queueing system with waiting and with the simplest input flow (we use the terminology of A. Ya. Khinchin ([1])). It had previously been established ([2]) that if in such a system the service duration is exponentially distributed, then the outgoing flow will be the simplest one. It follows from this that, in such a situation, finding the mathematical expectation of the service duration from the totality of multidimensional distributions characterizing the outgoing flow appears impossible.

In the present note we show that this case is exceptional: if only the load of the system is less than critical and the service duration has a distribution law different from the exponential one, then this law is uniquely recovered from the joint distribution of two successive intervals between the completion of service of demands.

2. Consider a single-server queueing system with waiting, with the simplest input flow of intensity λ, and with service durations represented by independent, jointly distributed random variables η_n with common distribution function $F(x)$. Denote $M\eta_1 = \tau$. Let, further, ξ_n be the duration of the interval between the departures from the system of the n-th and $(n+1)$-st demands;

$$\Phi(x) = \lim_{n\to\infty} \mathbf{P}\{\xi_n < x\}; \qquad \rho = \lambda\tau; \qquad \alpha(s) = \int_0^\infty e^{-sx}\, d\Phi(x); \qquad \Phi_1(x,y) =$$

$$= \lim_{n\to\infty} \mathbf{P}\{\xi_n < x,\ \xi_{n+1} < y\}; \qquad \alpha_1(s,t) = \int_0^\infty \int_0^\infty e^{-sx-ty}\, d\Phi_1(x,y);$$

$$\psi(s) = \int_0^\infty e^{-sx}\, dF(x).$$

The introduced limits exist, since for $\rho < 1$ the number of demands in the system at any time t is a regenerating random process, of which the quantities ξ_n are also functionals; whereas if $\rho \geqslant 1$, then, by a known result of D. Lindley ([3]), the probability of the equality $\xi_n = \eta_{n+1}$ tends to unity as $n \to \infty$.

Theorem. If $\rho < 1$, the formula

$$\psi(s) = \frac{\lambda + s}{\lambda + \rho s}\, \alpha(s), \tag{1}$$

is valid.

The constant ρ is determined as the unique constant different from 1 for which, identically with respect to $s, t \geqslant 0$,

$$\alpha_1(s,t) = \alpha(t) \frac{\lambda + t}{\lambda + \rho t} \left[\alpha(s) - (1-\rho)\alpha(s+\lambda) \frac{2\lambda + s}{\lambda + \lambda\rho + \rho s} \times \right.$$
$$\left. \times \left(\frac{1+\rho}{2\alpha(\lambda)} - \frac{s}{s+\lambda} \right) \frac{t}{\lambda + t} \right]. \tag{2}$$

The constant λ is found from the condition

$$\lambda^{-1} = -\alpha'(0). \tag{3}$$

For $\rho \geqslant 1$ we have

$$\alpha_1(s,t) = \psi(s)\psi(t); \tag{4}$$

in this case identity (2) is valid only when ρ is replaced by one.

3. **Proof.** Let first $\rho < 1$. We shall assume that the distribution of the random variable ζ_n coincides with $\Phi(x)$ (for this it suffices to specify the corresponding initial distribution). Then, with probability ρ, $\zeta_n = \eta_{n+1}$ (this equality will hold in the case when the $(n+1)$-st demand finds the servicing device free); with probability $1-\rho$, ζ_n will consist of two independent intervals: η_{n+1} and an exponentially distributed interval between the departure from the system of the n-th demand and the arrival of the $(n+1)$-st demand. The formula of total probability leads to the equality

$$\Phi(x) = \rho F(x) + (1-\rho)\int_0^x [1 - e^{-\lambda(x-y)}]\, dF(y); \tag{5}$$

the latter is, obviously, equivalent to formula (1). From this, (3) also follows immediately.

Introduce the auxiliary random variables: ν_n is the number of demands remaining in the system after completion of service of the n-th demand; μ_n is the number of demands arriving into the system during the service time of the n-th demand; ξ_n is the time from the moment of completion of service of the n-th demand to the moment of arrival of the next demand; $I(A)$ is the indicator of the event A ([4]).

The argument used in deriving equality (5) may be symbolically written as follows:

$$\zeta_n = \eta_{n+1} + \xi_n I(\nu_n = 0). \tag{6}$$

Furthermore, $\nu_{n+1} = 0$ if and only if $\nu_n \leqslant 1$ and $\mu_{n+1} = 0$, whence

$$\xi_{n+1} = \eta_{n+2} + \xi_{n+1} I(\nu_n \leqslant 1,\ \mu_{n+1} = 0). \tag{7}$$

In equalities (6) and (7) the random variable ν_n has the distribution coinciding with the stationary distribution of the number of demands in the system ([1]); the latter is given by the well-known Pollaczek-Khinchin formula. From this formula we find:

$$\mathbf{P}\{\nu_n = 0\} = 1-\rho, \qquad \mathbf{P}\{\nu_n \leqslant 1\} = (1-\rho)/\alpha(\lambda).$$

Then, η_{n+1} and η_{n+2} are independent; ξ_{n+1} and ξ_{n+2} do not depend on the random vector $\{\nu_n, \eta_{n+1}, \eta_{n+2}\}$. Finally, μ_{n+1} depends only on η_{n+1}; for $\eta_{n+1} = x$ the conditional probability of the equality $\mu_{n+1} = 0$ is $e^{-\lambda x}$. Using the indicated properties and applying the expectation operator to both sides of the equality

$$\exp\{-s\xi_n - t\xi_{n+1}\} =$$

$$= \exp\{-s[\eta_{n+1} + \xi_n I(\nu_n = 0)] - t[\eta_{n+2} + \xi_{n+1} I(\nu_n \leqslant 1,\ \mu_{n+1} = 0)]\},$$

we find the formula

$$\alpha_1(s,t) = \psi(t)\left[\frac{\lambda+\rho s}{\lambda+s}\psi(s) - (1-\rho)\left(\frac{1}{\psi(\lambda)} - \frac{s}{\lambda+s}\right)\frac{t}{\lambda+t}\psi(s+\lambda)\right].$$

In order to obtain (2) from it, it suffices to replace $\psi(s)$ by $\dfrac{\lambda+s}{\lambda+\rho s}\alpha(s)$.

Suppose that the identity (2) is satisfied when ρ is replaced by some constant $r \neq 1$ different from it. Then we shall have the identity

$$(\lambda{+}rt)\left[(\lambda+t)\alpha(s)-(1-\rho)t\alpha(s+\lambda)(2\lambda+s)(\lambda+\lambda\rho+\rho s)^{-1}\left(\frac{1+\rho}{2\alpha(\lambda)}-\frac{s}{s+\lambda}\right)\right]$$

$$\equiv(\lambda+\rho t)\,[(\lambda+t)\alpha(s)-(1-r)t\alpha(s+\lambda)\times$$

$$\times(2\lambda+s)(\lambda+\lambda r+rs)^{-1}\left(\frac{1+r}{2\alpha(\lambda)}-\frac{s}{s+\lambda}\right)\Bigg].$$

Equating the coefficients of t and t^2 on both sides, we obtain that $2\alpha(\lambda)=1$ and the function $\alpha(s)$ must satisfy the functional equation

$$\alpha(s)(\lambda+s)=\alpha(s+\lambda)(2\lambda+s). \tag{8}$$

Note that, by virtue of (8), $\alpha(s)$ is analytically continued to the whole left half-plane; moreover, owing to the periodicity of the function $\alpha(s)(\lambda+s)$ and the inequality $|\alpha(s)|\leqslant 1$, $\operatorname{Re} s\geqslant 0$, $\alpha(s)$ is bounded everywhere outside a neighborhood of the point $s=-\lambda$. Application of the generalized Liouville theorem ([5]) leads us to the conclusion that $\alpha(s)=\lambda/(\lambda+s)$, i.e., that the service time is distributed according to the exponential law. The part of the theorem that pertains to the case $\rho\geqslant 1$ is trivial and does not require a special proof.

4. From the theorem proved it follows that, for fixed $\alpha(s)$, there exists a linear functional of the distribution $\Phi_1(x,y)$ by which ρ is uniquely recovered. However, the statistic associated with a functional suitable for all $\alpha(s)$ apparently has a rather complicated form. For practical purposes one may recommend the equations obtained by comparing derivatives of both sides of equality (2) at $s=t=0$. Thus, the simplest of these equations has the form

$$(2\lambda\alpha_1-\alpha_0)\rho^2+2\alpha_0[\lambda^2m+2\alpha_0-2]\rho+2\lambda^2\alpha_0m-2\lambda\alpha_1-4\alpha_0^2+\alpha_0=0,$$

where

$$\alpha_0=\alpha(\lambda),\qquad \alpha_1=-\alpha'(\lambda),\qquad m=\{\partial^2\alpha_1(s,t)/\partial s\,\partial t\}_{s=t=0}.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.