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Abstract

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ON AN ANALOGUE OF THE PLEMELJ-PRIVALOV THEOREM IN THE CASE OF NONSMOOTH CURVES AND ITS APPLICATIONS

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Let a closed rectifiable Jordan curve Γ satisfy the metric relation

$$s(t_1, t_2) \leq \beta(|t_1 - t_2|), \tag{1}$$

where $s(t_1, t_2)$ is the smaller of the two arcs joining the points t_1 and t_2, and the function $\beta(\delta)$, $0 \leq \delta \leq l_0$ (l_0 is the diameter of the curve Γ), satisfies the conditions: 1) $\beta(\delta)$ is a continuous, increasing function; 2) $\beta(0) = 0$; 3) $\beta(\delta)/\delta$ is almost decreasing, i.e., there exists a constant $k > 0$ such that for $\delta_1 > \delta_2$,

$$\beta(\delta_1)/\delta_1 \leq k\,\beta(\delta_2)/\delta_2.$$

By $\alpha(\delta)$ we denote the function inverse to $\beta(\delta)$. Introduce the modulus of continuity of the function $f(t)$, defined on Γ:

$$\omega(f, \delta) = \sup_{|t_1 - t_2| \leq \delta} |f(t_1) - f(t_2)|, \qquad 0 < \delta \leq l_0.$$

Consider the integral

$$\int_\Gamma \frac{f(t)}{t - t_0}\, dt = \lim_{\varepsilon \to 0} \int_{\Gamma - \Gamma_\varepsilon(t_0)} \frac{f(t)}{t - t_0}\, dt, \tag{2}$$

where $\Gamma_\varepsilon(t_0)$ denotes the part of the curve Γ containing the point $t_0 = t(s_0)$ and having endpoints $t(s_0 - \varepsilon)$, $t(s_0 + \varepsilon)$.

Consider the class Φ of functions $\varphi(\delta)$, defined on $[0, l/2]$ and satisfying the conditions: 1) $\varphi(\delta)$ is continuous and monotonically increasing on $[0, l/2]$; 2) $\varphi(\delta) \neq 0$ for every $\delta \in [0, l/2]$, $\varphi(0) = 0$; 3) $\varphi(\delta)/\delta$ is almost decreasing.

Let $\varphi(\delta) \in \Phi$. A function $f(t)$, defined on Γ, is said to belong to the class H_φ if there exists a constant $c > 0$ such that, for $0 < \delta \leq l_0$,

$$\omega(f, \delta) \leq c\varphi(\delta).$$

If in H_φ one introduces the norm

$$\|f\|_{H_\varphi} = \max_{t \in \Gamma} |f(t_0)| + \sup_{0<\delta\leq l_0} (\omega(f,\delta)/\varphi(\delta)),$$

then, as is not difficult to see, H_φ becomes a Banach space.

Theorem 1. Let a closed rectifiable Jordan curve Γ satisfy condition (1) and

$$\int_\Gamma \frac{dt}{t-t_0} = \pi i.$$

If a function $f(t) \in H_\varphi$, defined on Γ, is such that

$$\int_0^{l/2} \frac{\varphi(\alpha(s))}{\alpha(s)}\, ds < +\infty,$$

then for the function

$$g(t_0) = \int_\Gamma \frac{f(t)}{t-t_0}\, dt$$

the inequality

$$\omega(g,\delta) \leq c\|f\| \left[\int_0^{\beta(\delta)} \frac{\varphi(\alpha(s))}{\alpha(s)}\, ds + \delta \int_{\beta(\delta)}^{l/2} \frac{\varphi(\alpha(s))}{\alpha(s)}\, ds \right] \tag{3}$$

holds for $0 < \delta \leq \delta_0$, where the constants depend on Γ and φ; δ_0 depends only on the curve Γ (l is the length of the curve Γ).

Theorem 2 ([3]). Let $\varphi(\delta), \psi(\delta) \in \Phi$.

1) If

$$0 < \lim_{\delta\to 0} \frac{\varphi(\delta)}{\psi(\delta)} \leq \lim_{\delta\to 0} \frac{\varphi(\delta)}{\psi(\delta)} < +\infty,$$

then the classes H_φ and H_ψ coincide, and the norms are equivalent.

2) If

$$\lim_{\delta\to 0} \frac{\varphi(\delta)}{\psi(\delta)} = 0,$$

then H_φ is a proper part of H_ψ.

By definition, a function $\varphi(\delta) \in \psi(\beta)$ if $\varphi(\delta) \in \Phi$ and the relation

$$\int_0^{\beta(\delta)} \frac{\varphi(\alpha(s))}{\alpha(s)}\, ds = O\left[\frac{\varphi(\delta)\beta(\delta)}{\delta}\right]. \tag{z}$$

is satisfied.

We also consider the class $\psi_1(\beta)$ of functions $\varphi(\delta) \in \Phi$ such that

$$\int_{\beta(\delta)}^{l/2} \frac{\varphi(\alpha(s))}{\alpha^2(s)}\, ds = O\left[\frac{\varphi(\delta)\beta(\delta)}{\delta^2}\right]. \qquad (z_1)$$

From the analysis in the work ([4]) it follows:

Lemma. 1) If $\varphi(\delta) \in \psi(\beta)$, then condition (z) is equivalent to each of the following two:

(I) there exists a constant $c > 1$ such that

$$\lim_{\delta\to 0} \frac{\varphi(\alpha(c\delta))\alpha(\delta)}{\varphi(\alpha(\delta))\alpha(c\delta)} > \frac{1}{c}.$$

(s) there exists a constant a $(0 < a < 1)$ such that the function

$$\frac{\varphi(\delta)}{\delta}[\beta(\delta)]^{1-a}$$

is almost increasing on $[0, l/2]$.

2) Similarly, if $\varphi(\delta) \in \psi_1(\beta)$, then condition (z_1) is equivalent to each of the following:

(I_1). There exists a constant $c > 1$ such that

$$\lim_{\delta\to 0} \frac{\varphi(\alpha(c\delta))\alpha^2(\delta)}{\varphi(\alpha(\delta))\alpha^2(\delta)} < \frac{1}{c}.$$

(s_1) There exists a constant $\alpha > 0$ such that the function

$$\frac{\varphi(\delta)}{\delta^2}[\beta(\delta)]^{1+\alpha}$$

is almost decreasing.

Theorem 3. Let the closed rectifiable Jordan curve Γ satisfy condition (1) and

$$\int_\Gamma \frac{dt}{t - t_0} = \pi i;$$

let $\varphi(\delta) \in \psi(\beta) \cap \psi_1(\beta)$.

Then, if $\omega(f, \delta) = O[\varphi(\delta)]$, then

$$\omega(g, \delta) = O\left[\varphi(\delta)\frac{\beta(\delta)}{\delta}\right],$$

where

$$g(t_0) = \int_\Gamma \frac{f(t)}{t - t_0}\, dt \quad (t_0 \in \Gamma).$$

It is not difficult to formulate Theorem 3 in terms of the operator

$$Af = \int_\Gamma \frac{f(t)}{t - t_0}\, dt.$$

Corollary 1. If the conditions of Theorem 3 are satisfied, then the operator A acts from H_φ into H_ψ, where

$$\psi(\delta) = \varphi(\delta)\frac{\beta(\delta)}{\delta},$$

and is bounded.

Corollary 2. If Γ is a smooth curve, $\varphi(\delta) \in \Phi$, and there exists a constant $c > 1$ such that

$$1 < \varliminf_{\delta\to 0} \frac{\varphi(c\delta)}{\varphi(\delta)} \leqslant \varlimsup_{\delta\to 0} \frac{\varphi(c\delta)}{\varphi(\delta)} < c,$$

then the operator A acts in H_φ and is bounded.

This assertion, which is a direct generalization of the Plemelj-Privalov theorem, was proved in ([3]).

Corollary 3. If $\beta(\delta) = \text{const}\ \delta^\gamma$, $0 < \gamma < 1$, then the operator A acts boundedly from H_α into $H_{\alpha-(1-\gamma)}$ for $1-\gamma < \alpha$, where H_α $(0 < \alpha < 1)$ denotes H_{δ^α}.

Using the theorem of N. A. Davydov ([5]) on boundary values of the Cauchy integral, Theorem 3, and the method of solving the Riemann problem ([2,6]), one can prove the following theorem.

Theorem 4. Suppose that for a closed rectifiable Jordan curve Γ condition (1) is satisfied and

$$\int_\Gamma \frac{dt}{t - t_0} = \pi i, \qquad \text{where } \beta(\delta) = \text{const}\ \delta^\gamma, \quad 1/2 < \gamma \leqslant 1.$$

Then for the Riemann problem

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t),$$

where $G(t) \neq 0$, $t \in \Gamma$, $G(t) \in H_{2(1-\gamma)+\varepsilon}$, $g(t) \in H_{(1-\gamma)+\varepsilon}$, $\varepsilon > 0$, all the theorems proved in ([6]) are valid.

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Note: Figure translations are in progress. See original paper for figures.