# ON SCATTERING OPERATORS AND SEMIGROUPS OF CONTRACTIONS IN A HILBERT SPACE

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## Abstract

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*MATHEMATICS*

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# ON SCATTERING OPERATORS AND SEMIGROUPS OF CONTRACTIONS IN A HILBERT SPACE

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1. Let $\mathfrak{D}_+$ and $\mathfrak{D}_-$ be subspaces belonging to the intersection of Hilbert spaces $\mathfrak{H}$ and $\widetilde{\mathfrak{H}}$. Here on $\mathfrak{D}_+$ and $\mathfrak{D}_-$ (but not on their linear sum) the Hilbert metrics of the spaces $\mathfrak{H}$ and $\widetilde{\mathfrak{H}}$ coincide. Let $U_t$ and $\widetilde{U}_t$ be groups of unitary operators in $\mathfrak{H}$ and $\widetilde{\mathfrak{H}}$, respectively, satisfying the conditions:

$$1) \quad U_{\pm t}\mathfrak{D}_\pm \subset \mathfrak{D}_\pm, \quad t > 0; \qquad 2) \quad U_{\pm t} f = \widetilde{U}_{\pm t} f, \quad f \in \mathfrak{D}_\pm, \quad t > 0.$$

Here discrete groups (the parameter $t$ takes only integral values) and continuous groups ($-\infty < t < \infty$) are considered in parallel. By $\mathfrak{H}_\pm$ denote the l.c.s. $\{U_t\mathfrak{D}_\pm\}_{-\infty}^{+\infty}$, and by $\widetilde{\mathfrak{H}}_\pm$ the analogous subspaces for $\widetilde{U}_t$. Without loss of generality, assume that

$$\bigcap_t U_t\mathfrak{D}_\pm = \bigcap_t \widetilde{U}_t\mathfrak{D}_\pm = \{0\}, \qquad \mathfrak{D}_+ \cap \mathfrak{D}_- = \{0\}; \tag{1}$$

$$\overline{\mathfrak{H}_+ + \mathfrak{H}_-} = \mathfrak{H}, \qquad \overline{\widetilde{\mathfrak{H}}_+ + \widetilde{\mathfrak{H}}_-} = \widetilde{\mathfrak{H}}. \tag{2}$$

In paper ([1]) the groups $U_t$ and $\widetilde{U}_t$ were studied from the point of view of scattering theory within the framework of the scheme of P. Lax and R. Phillips ([2]), i.e. it was assumed that the subspaces $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are orthogonal both in $\mathfrak{H}$ and in $\widetilde{\mathfrak{H}}$, $\mathfrak{H}_\pm = \mathfrak{H}$, and $\widetilde{\mathfrak{H}}_\pm = \widetilde{\mathfrak{H}}$. Below, in particular, it is clarified how the orthogonality of the subspaces $\mathfrak{D}_+$ and $\mathfrak{D}_-$ is connected with the analytic properties of the scattering suboperator. The more general scheme adopted in this article makes it possible to study groups of unitary operators that are dilations of arbitrary semigroups of contractions ([3]).

Define the operators $W_\pm$, acting from $\mathfrak{H}_\pm$ into $\widetilde{\mathfrak{H}}_\pm$, by setting, for $f \in \mathfrak{H}_\pm$,

$$W_\pm(\widetilde{U}, U)f = s\text{-}\lim_{T\to\pm\infty} s\text{-}\lim_{t\to\pm\infty} \widetilde{U}_{-t} U_{t-T} P_{\mathfrak{D}_\pm} U_{Tf}. \tag{3}$$

Here, as in what follows, $P_{\mathfrak{R}}$ is the orthogonal projector onto the subspace $\mathfrak{R}$, and $s$-lim is the strong limit. The operators $W_\pm$ isometrically map $\mathfrak{H}_\pm$ onto $\widetilde{\mathfrak{H}}_\pm$.

We extend them to $\mathfrak{H}_\pm^\perp = \mathfrak{H} \ominus \mathfrak{H}_\pm$ by the condition $W_\pm \mathfrak{H}_\pm^\perp = 0$ and call them wave operators. The scattering operator of the group $\widetilde{U}_t$ with respect to $U_t$ is defined as the product

$$S(\widetilde{U}, U) = W_+^*(\widetilde{U}, U) W_-(\widetilde{U}, U) (= W_+(U, \widetilde{U}) W_-(\widetilde{U}, U)). \tag{4}$$

It is easy to see that

$$S\mathfrak{H} \subset \mathfrak{H}_+, \qquad S^*\mathfrak{H} \subset \mathfrak{H}_-, \qquad U_{tS} = SU_t, \quad \|S\| \leq 1. \tag{5}$$

Note that the operator $S$ is unitary if and only if $\mathfrak{H}_\pm = \mathfrak{H}$ and $\widetilde{\mathfrak{H}}_\pm = \widetilde{\mathfrak{H}}$. Below we shall consider only the part $S_+$ of the operator $S$ on $\mathfrak{H}_+$ and the part $S_-$ of the operator $S^*$ on $\mathfrak{H}_-$.

In what follows $U$ denotes the operator $U_1$, if $U_t$ is a discrete group, and the Cayley transform $U = (A + iI)(A - iI)^{-1}$ of the infinitesimal ope-

operator $A = s\text{-}\lim_{t\downarrow 0} \dfrac{1}{-it}(U_t - I)$, if $U_t$ is a continuous group; $\mathfrak{N}_\pm$ are subspaces $\mathfrak{D}_\pm \ominus U^{\pm 1}\mathfrak{D}_\pm$; $\Lambda$ is the interval $(-\pi, \pi)$ and $(-\infty, \infty)$, respectively, in the discrete and continuous cases; $C_\pm$ are the interior and exterior of the unit circle or the upper and lower half-planes of the complex plane; $\mathcal{L}_2(\mathfrak{N}_\pm, \Lambda)$ are Hilbert spaces of measurable vector-functions $f(\lambda)$ with values in $\mathfrak{N}_\pm$ and finite norm

$$\|f\|_{L_2}^2 = \frac{1}{2\pi} \int_\Lambda \|f(\lambda)\|_{\mathfrak{N}_\pm}^1 \, d\lambda < \infty;$$

$\mathcal{L}_2^\pm(\mathfrak{N}_\pm, \Lambda)$ are subspaces of vector-functions from $\mathcal{L}_2(\mathfrak{N}_\pm, \Lambda)$ that are boundary values of vector-functions, analytic in $C_\pm$, of the Hardy classes $H_2^\pm(\mathfrak{N}_\pm)$.

From condition (1) and the definition of the spaces $\mathfrak{H}_\pm$ there follows the existence of isometric mappings $F_\pm$ of the spaces $\mathfrak{H}_\pm$ onto $\mathcal{L}_2(\mathfrak{N}_\pm, \Lambda)$ such that

$$F_\pm U_t F_\pm^{-1} f(\lambda) = e^{it\lambda} f(\lambda),$$

$$F_\pm \mathfrak{D}_\pm = \mathcal{L}_2^\pm(\mathfrak{N}_\pm, \Lambda).$$

By virtue of (5), there exist measurable almost everywhere nonexpanding operator-functions $S_\pm(\lambda)$, acting for each $\lambda \in \Lambda$ in $\mathfrak{N}_\pm$, such that

$$F_\pm S_\pm F_\pm^{-1} f(\lambda) = S_\pm(\lambda) f(\lambda), \qquad f(\lambda) \in \mathcal{L}_2(\mathfrak{N}_\pm, \Lambda).$$

The operators $S_\pm(\lambda)$ are unitary almost everywhere if and only if $\mathfrak{H}_+ = \mathfrak{H}_-$ and $\widetilde{\mathfrak{H}}_+ = \widetilde{\mathfrak{H}}_-$; in this case the groups $U_t$ and $\widetilde{U}_t$ have only Lebesgue spectrum on $\Lambda$ of multiplicity $r = \dim \mathfrak{N}_+ = \dim \mathfrak{N}_-$. In the general case the groups $U_t$

and $\widetilde{U}_t$ have absolutely continuous, but not necessarily Lebesgue, spectrum [3]; however, their parts considered respectively on $\mathfrak{H}_\pm$ and $\widetilde{\mathfrak{H}}_\pm$ have only Lebesgue spectrum of multiplicity $r_\pm = \dim \mathfrak{N}_\pm$.

2. Of interest are the cases when the operator-functions $S_\pm(\lambda)$ are boundary values of analytic functions. From the results of P. Lax [4] it follows

**Theorem 1.** *In order that the operator-functions $S_\pm(\lambda)$ be boundary values (in the strong sense) of analytic operator-functions $S_\pm(\zeta)$:*

a) *on $C_+$, it is necessary and sufficient that $S_+\mathfrak{D}_+ \subset \mathfrak{D}_+$;*

b) *on $C_-$, it is necessary and sufficient that $S_\pm(\mathfrak{H}_\pm \ominus \mathfrak{D}_\pm) \subset \mathfrak{H}_\pm \ominus \mathfrak{D}_\pm$.*

According to (1), the subspaces $\mathfrak{D}_+$ and $\mathfrak{D}_-$ intersect only in zero. Therefore one can construct the space $\mathfrak{H}_0 = \mathfrak{D}_+ \oplus \mathfrak{D}_-$. For definiteness in what follows, we shall assume that $r_+ \geq r_-$. Then there exists a subspace $\mathfrak{D}'_+ \subset \mathfrak{D}_+$, for which $U_t\mathfrak{D}'_+ \subset \mathfrak{D}'_+$ and $r'_+ = \dim \mathfrak{N}'_+ = r_-$ ($\mathfrak{N}'_+ = \mathfrak{D}'_+ \ominus U\mathfrak{D}'_+$). In the role of the group $\widetilde{U}_t$ henceforth will appear the group $U'_t$ in the space $\mathfrak{H}'_0 = \mathfrak{D}'_+ \oplus \mathfrak{D}_-$. We note that the group $U'_t$ is determined uniquely up to an isometric mapping of $\mathfrak{N}_-$ onto $\mathfrak{N}'_+$. For the corresponding operator-functions $S_\pm(\lambda)$ the following is valid

**Theorem 2.** *In order that the operator-functions $S_\pm(\lambda)$ be boundary values of operator-functions analytic in $C_\pm$, it is necessary and sufficient that the subspaces $\mathfrak{D}'_+$ and $\mathfrak{D}_-$ be orthogonal in $\mathfrak{H}$.* *

When the condition of Theorem 2 is fulfilled, in the case of discrete groups

$$\begin{aligned} S_+(\xi) &= U^0 P_{\mathfrak{N}_-}(U - \xi I)^{-1} P_{\mathfrak{N}'_+}, && |\xi| < 1; \\ S_-(\xi) &= U^{0-1} P_{\mathfrak{N}'_+}(U^{-1} - \xi^{-1} I)^{-1} P_{\mathfrak{N}_-}, && |\xi| > 1. \end{aligned} \tag{6}$$

* In the general case, if $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are orthogonal both in $\mathfrak{H}$ and in $\widetilde{\mathfrak{H}}$, for the suboperators $S_\pm(\lambda)$ of the operators $S_\pm = S_\pm(\widetilde{U}, U)$ there remains valid a factorization theorem analogous to that formulated in [4].

For continuous groups, from (6) we obtain

$$\begin{aligned} S_+(\zeta) &= -\frac{(\zeta - i)^2}{2i} U_0 P_{\mathfrak{N}_-}(A - \zeta I)^{-1} P_{\mathfrak{N}'_+}, && \operatorname{Im}\zeta < 0; \\ S_-(\zeta) &= \frac{(\zeta + i)^2}{2i} U_0^{-1} P_{\mathfrak{N}'_+}(A - \zeta I)^{-1} P_{\mathfrak{N}_-}, && \operatorname{Im}\zeta > 0. \end{aligned} \tag{7}$$

**3.** Suppose that $\mathfrak{H}_0 \subset \mathfrak{H}$, and consider in $\mathfrak{K} = \mathfrak{H} \ominus \mathfrak{H}_0$ the one-parameter family of operators $T_t$ defined by the formula

$$T_t = P_{\mathfrak{K}} U_t P_{\mathfrak{K}}, \qquad t > 0. \tag{8}$$

The family $T_t$ forms a nonexpanding semigroup ([2]). It is easy to see that the group $U_t$ is a unitary dilation in the sense of Nagy ([3]) of the semigroup $T_t$, where property (1) is necessary and sufficient for the minimality of the dilation (l.c.s. $\{U_t\mathfrak{K}\}_{-\infty}^{+\infty} = \mathfrak{H}$), and property (2) is necessary and sufficient for the complete nonunitarity of the semigroup $T_t$ (absence of a unitary part).

The condition $\mathfrak{H}_+ = \mathfrak{H}$ ($\mathfrak{H}_- = \mathfrak{H}$) is equivalent to the requirement $s\text{-}\lim_{t\to\infty} T_t = 0$ ($s\text{-}\lim_{t\to\infty} T_t^* = 0$). Let us note that if $Z$ is the Cayley transform of the infinitesimal operator of the semigroup $T_t$, then the condition $s\text{-}\lim_{t\to\infty} T_t = 0$ is equivalent to the condition $s\text{-}\lim_{n\to\infty} Z^n = 0$. This is a consequence of the equality

$$s\text{-}\lim_{t\to\infty} T_t^* T_t = s\text{-}\lim_{n\to\infty} Z^{*n} Z^n. \tag{9}$$

Let now a nonexpanding semigroup $T_t$ be defined in some Hilbert space $\mathfrak{K}$, and let $U_t$ be some unitary dilation of it with range in the space $\mathfrak{H}$.

**Theorem 3.** *The space $\mathfrak{H}$ is representable in the form*

$$\mathfrak{H} = \mathfrak{D}_+ \oplus \mathfrak{K} \oplus \mathfrak{D}_-, \tag{10}$$

*where $\mathfrak{D}_\pm$ are subspaces of the space $\mathfrak{H}$ possessing the property $U_{\pm t}\mathfrak{D}_\pm \subset \mathfrak{D}_\pm$, $t > 0$. In the case of minimality of the dilation $U_t$ (and only then) the representation (10) is unique.*

The construction of the representation (10) for the minimal dilation of a discrete group is in fact contained in ([3]), where

$$\mathfrak{D}_\pm = \text{l.c.s. } \{(I - P_{\mathfrak{K}}) \cdot U_{\pm t}\mathfrak{K}\}_{t>0}.$$

**4.** For a semigroup of nonexpanding operators $T_t$ in a Hilbert space $\mathfrak{K}$, denote by $T$ the operator $T_1$, if $T_t$ is a discrete semigroup, and the Cayley transform $T = (B + iI)(B - iI)^{-1}$ of the infinitesimal operator

$$B = s\text{-}\lim_{t\downarrow 0} \frac{1}{-it}(T_t - I),$$

if $T_t$ is a continuous semigroup; by $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$ denote the subspaces

$$\mathfrak{D}_T = \overline{(I - T^*T)^{1/2}\mathfrak{K}}, \qquad \mathfrak{D}_{T^*} = \overline{(I - TT^*)^{1/2}\mathfrak{K}}.$$

The characteristic function of a nonexpanding operator $T$ (of a discrete semigroup of nonexpanding operators $T_t$) is the function whose values are operators acting from $\mathfrak{D}_T$ to $\mathfrak{D}_{T^*}$ according to the formula

$$W_T(z) = -T + z(I - TT^*)^{1/2}(I - zT^*)^{-1}(I - T^*T)^{1/2}. \tag{11}$$

The characteristic function of a continuous semigroup $T_t$ (of a dissipative operator $B$) is obtained from (11) by replacing the argument $z = (\zeta + i)/(\zeta - i)$. When the operator $B$ is unbounded, this definition of the characteristic function of a dissipative operator coincides with the generally accepted one ([5]).

Let $U_t$ be a minimal unitary dilation of the semigroup $T_t$ with exit into the space $\mathfrak{H} = \mathfrak{D}_+ \oplus \mathfrak{K} \oplus \mathfrak{D}_-$ and $\mathfrak{N}_\pm = \mathfrak{D}_\pm \ominus U^{\pm 1}\mathfrak{D}_\pm$. Then there exist isometric mappings $V_\pm$ of the spaces $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$ onto $\mathfrak{N}_+$ and $\mathfrak{N}_-$, respectively, (3), such that for $h \in \mathfrak{K}$

$$V_+(I - T^*T)^{1/2}h = (U - T)h, \qquad V_-(I - TT^*)^{1/2}h = (U^* - T^*)h.$$

Therefore, in particular, $r_+ = \dim \mathfrak{N}_+ = \dim \mathfrak{D}_T$, $r_- = \dim \mathfrak{N}_- = \dim \mathfrak{D}_{T^*}$. By Theorem 2, the scattering suboperators $S_\pm(\lambda)$ for $S_\pm = S_\pm(U^0, U)$ are the boundary values of operator functions $S_\pm(\zeta)$ analytic in $C_\pm$.

The following theorem was established by the authors already in (1) under the particular assumption that $T_t$ and $T_t^*$ converge strongly to zero as $t \to \infty$.

**Theorem 4.** *The operator functions $S_\pm(\zeta)$ are related to the characteristic functions $W_T(\zeta)$ and $W_{T^*}(\zeta)$ by the relations*

$$S_+(\zeta) = U^0 V_- W_T(\zeta) V_+^{-1}, \qquad \zeta \in C_+; \tag{12}$$

$$S_-(\zeta) = U^{0-1} P_{\mathfrak{N}'_+} V_+ W_{T^*}(\zeta) V_-^{-1}, \qquad \zeta \in C_-.$$

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*Note: Figure translations are in progress. See original paper for figures.*