# ON PROPERTIES OF THE DETERMINANT OF MATRICES COMMUTING WITH SIMILAR INTEGER MATRICES, AND SOME OF THEIR APPLICATIONS

MATHEMATICS

1965

SovietRxiv

## Abstract

## Full Text

UDC 512.83

**MATHEMATICS**

**B. A. ZEYNALOV**

# ON PROPERTIES OF THE DETERMINANT OF MATRICES COMMUTING WITH SIMILAR INTEGER MATRICES, AND SOME OF THEIR APPLICATIONS

*(Presented by Academician A. I. Mal' tsev on 12 IV 1965)*

The present note is devoted to an exposition of the question of the conditions for decomposability and indecomposability into linear factors of the determinant of the general solution of the matrix equation $AX = XB$, where $A$ and $B$ are two similar integer matrices of order $n$. In addition, two applications of the result obtained are indicated: 1) an application to the solution of the equation $AX = XB$ in integer unimodular matrices; 2) an application to the units of a cyclic algebraic number field of prime degree $p$.

1. If $A$ and $B$ are two similar integer matrices of order $n$, and $K$ is the quasi-natural normal form ($^1$) of these matrices over the field of rational numbers, then the general solutions $X$ and $S$ of the matrix equations

$$AX = XB, \tag{1}$$

$$KS = SK \tag{2}$$

over the field of rational numbers are connected by the equality

$$X = Q_0^{-1} S Q_0 T_0, \tag{3}$$

where $T_0$ and $Q_0$ are certain nonsingular matrices satisfying the equalities

$$AT_0 = T_0 B, \qquad KQ_0 = Q_0 A. \tag{4}$$

2. Let us indicate the form of the solution of the matrix equation (2). Let

$$[\varepsilon_i(\lambda)]^{k_{i1}}, \dots, [\varepsilon_i(\lambda)]^{k_{iq_i}} \quad (i = 1, \dots, t) \tag{5}$$

be the elementary divisors of the quasi-natural matrix $K$ over the field of rational numbers, where their exponents satisfy the inequalities

$$k_{i1} \leq k_{i2} \leq \cdots \leq k_{iq_i} \quad (i = 1, \dots, t). \tag{6}$$

Condition (2) reduces to equalities of the same type, of the form

$$K_1 U = U K_1, \tag{7}$$

where $K_1$ is a quasi-natural matrix with elementary divisors

$$[\varepsilon_1(\lambda)]^{k_{11}}, \dots, [\varepsilon_1(\lambda)]^{k_{1q_1}}, \tag{8}$$

$$k_{11} \leq k_{12} \leq \cdots \leq k_{1q_1}.$$

Condition (7) reduces to the equalities

$$A_{iU_{ij}} = U_{ij} A_j \quad (i, j = 1, \dots, q_1), \tag{9}$$

where $A_i$ and $A_j$ are the companion matrices, respectively, of the elementary divisors

$$f_i(\lambda) = [\varepsilon_1(\lambda)]^{k_{1i}}, \qquad f_j(\lambda) = [\varepsilon_1(\lambda)]^{k_{1j}},$$

and $U_{ij}$ are the required blocks of the corresponding sizes.

Denote by $\alpha$ and $\beta$ the orders of the blocks $A_i$ and $A_j$, and suppose that $i \geq j$.

The general solution of the equation $A_{iU_{ii}} = U_{ii} A_i$ can be represented in the form

$$U_{ii} = \begin{bmatrix} X_1 & E \\ X_1 & A_i \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ X_1 & A_i^{\alpha-1} \end{bmatrix}, \tag{10}$$

where $E$ is the identity matrix, and $X_1$ is an $\alpha$-dimensional vector with arbitrary coordinates.

The general solution of the equation $A_{iU_{ij}} = U_{ij} A_j$ can be represented in the form

$$U_{ij} = \begin{bmatrix} X_1 & E \\ X_1 & A_j \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ X_1 & A_j^{\alpha-1} \end{bmatrix}, \tag{11}$$

where $E$ is the identity matrix, and $X_1$ is a $\beta$-dimensional vector with arbitrary coordinates.

The general solution of the equation $A_{jU_{ji}} = U_{ji}A_i$ can be represented in the form

$$U_{ji} = \begin{bmatrix} X_1 & E \\ X_1 & A_i \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ X_1 & A_i^{\beta-1} \end{bmatrix}, \tag{12}$$

where $E$ is the identity matrix, $X_1$ is an $\alpha$-dimensional vector, the number of whose arbitrary coordinates is equal to $\beta$, and moreover $X_1$ is determined by the vector equation

$$X_1(f_j(A_i)) = \bar{0}. \tag{13}$$

After finding all the matrices $U_{ij}$ $(i, j = 1, 2, \dots, q)$, we obtain the general solution $U$ of equation (7).

The general solution $S$ of equation (2) is a quasidiagonal matrix, on the main diagonal of which there are $t$ matrices of the form $U$.

**3. Calculation of the determinant of the matrix $S$—the general solution of equation (2).** Denote by $p, q, \dots, l, k, m$ the degrees, and by

$$\begin{gathered} \lambda_1, \lambda_2, \dots, \lambda_p \\ \lambda_1, \lambda_2, \dots, \lambda_p, \dots, \lambda_q \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \lambda_1, \lambda_2, \dots, \lambda_p, \dots, \lambda_q, \dots, \lambda_m \end{gathered} \tag{14}$$

the roots of the polynomials (8).

It can be proved by calculation that the determinant of the matrix $U$, the general solution of equation (7), is equal to a product of the form

$$|U| = \prod_{i=1}^{p} \begin{vmatrix} F_{11}(\lambda_i) & F_{12}(\lambda_i) & \cdots & F_{1q_1}(\lambda_i) \\ F_{21}(\lambda_i) & F_{22}(\lambda_i) & \cdots & F_{2q_1}(\lambda_i) \\ \cdots & \cdots & \cdots & \cdots \\ F_{q_1 1}(\lambda_i) & F_{q_1 2}(\lambda_i) & \cdots & F_{q_1 q_1}(\lambda_i) \end{vmatrix} \cdot \prod_{i=p+1}^{q} \begin{vmatrix} F_{22}(\lambda_i) & \cdots & F_{2q_1}(\lambda_i) \\ \cdots & \cdots & \cdots \\ F_{q_1 2}(\lambda_i) & \cdots & F_{q_1 q_1}(\lambda_i) \end{vmatrix} \ldots$$

$$\ldots \prod_{i=l+1}^{k} \begin{vmatrix} F_{q_1-1,q_1-1}(\lambda_i) & F_{q_1-1,q_1}(\lambda_i) \\ F_{q_1,q_1-1}(\lambda_i) & F_{q_1 q_1}(\lambda_i) \end{vmatrix} \cdot \prod_{i=k+1}^{m} F_{q_1 q_1}(\lambda_i), \tag{15}$$

where

$$\begin{aligned} F_{i1}(\lambda) &= x_{i1} + \lambda x_{i2} + \ldots + \lambda^{p-1} x_{ip}, \\ F_{i2}(\lambda) &= y_{i1} + \lambda y_{i2} + \ldots + \lambda^{q-1} y_{iq}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ F_{iq_1}(\lambda) &= u_{i1} + \lambda u_{i2} + \ldots + \lambda^{m-1} u_{im} \quad (i = 1, \ldots, q_1), \end{aligned}$$

$x_{ij}, y_{is}, \ldots, u_{it}$ are parameters entering into the general solution $U$ of equation (7).

The determinant of the quasidiagonal matrix $S$ will be equal to several products of the form (15).

## 4. Calculation of the determinant of the matrix $S$—the general solution of equation (2) in the case when the exponents of the elementary divisors satisfy the inequalities

$$k_{i1} < k_{i2} < \ldots < k_{iq_i} \quad (i = 1, \ldots, t). \tag{16}$$

If the exponents of the elementary divisors (8) satisfy the inequalities

$$k_{11} < k_{12} < \ldots < k_{1q_1}, \tag{17}$$

then one can prove that the equalities

$$F_{\mu\nu}(\lambda_i) = 0 \quad (i = 1, 2, \ldots, m) \tag{18}$$

will hold for all $\mu$ and $\nu$ satisfying the inequalities

$$1 \leq \mu < \nu \leq q_1. \tag{19}$$

Denote by $r$ the degree, and by $\lambda_1, \ldots, \lambda_r$ the roots of the irreducible polynomial $\varepsilon_1(\lambda)$.

Using formula (15) and taking into account the preceding remark, by calculation we obtain

$$|U| = \left[\prod_{i=1}^{r} F_{11}(\lambda_i)\right]^{k_{11}} \cdot \left[\prod_{i=1}^{r} F_{22}(\lambda_i)\right]^{k_{12}} \cdots \left[\prod_{i=1}^{r} F_{q_1 q_1}(\lambda_i)\right]^{k_{1 q_1}}, \tag{20}$$

where

$$\begin{aligned} F_{11}(\lambda) &= x_{11} + \lambda x_{12} + \ldots + \lambda^{p-1} x_{1p}, \\ F_{22}(\lambda) &= y_{21} + \lambda y_{22} + \ldots + \lambda^{q-1} y_{2q}, \\ &\quad . \quad . \quad . \quad . \quad . \quad . \\ F_{q_1 q_1}(\lambda) &= u_{q_1 1} + \lambda u_{q_1 2} + \ldots + \lambda^{m-1} u_{q_1 m}. \end{aligned}$$

If the exponents of the elementary divisors of the matrix $K$ satisfy the inequalities (16), then we arrive at the conclusion that the determinant of the matrix $S$—the general solution of equation (2)—will be equal to several products of the form (20).

Let us note, in particular, that if the characteristic polynomial of the matrix $K$ of order $n$ is irreducible over the field of rational numbers, then the determinant of the matrix $S$—the general solution of equation (2)—is equal to

$$|S| = \prod_{i=1}^{n} F(\lambda_i), \tag{21}$$

where $F(\lambda) = x_1 + \lambda x_2 + \ldots + \lambda^{n-1} x_n$, $\lambda_i$ $(i = 1, \ldots, n)$ are the roots of the polynomial $\varphi(\lambda)$.

5. If among the exponents of the elementary divisors (8) there are equal ones, then the equalities (18) will not hold for all $\mu$ and $\nu$ satisfying the inequalities (19). Therefore the determinant of the matrix $U$—the general solution of equation (7) (and, consequently, also the determinant of the matrix $S$—the general solution of equation (2)) will not decompose entirely into linear factors.

Consider an example. Let the quasiscalar matrix $K$ of order 8 over the field of rational numbers have elementary divisors

$$\lambda^2 - p\lambda - q, \qquad \lambda^2 - p\lambda - q, \qquad (\lambda^2 - p\lambda - q)^2.$$

The determinant of the matrix $S$—the general solution of the equation $KS = SK$ —has the form

$$|S| = \prod_{i=1}^{2}\{[(x_1y_1 + qx_2y_2 - u_1v_1 - qu_2v_2) + \lambda_i(x_1y_2 + x_2y_1 + px_2y_2$$

$$-u_1v_2 - u_2v_1 - pu_2v_2)]\,[(z_1 + qz_3 + pqz_4) + \lambda_i(z_2 + pz_3 + (p^2 + q)z_4)]\},$$

where $\lambda_i$ $(i = 1, 2)$ are the roots of the polynomial $\lambda^2 - p\lambda - q$; $x_i,\ y_i,\ u_i,\ v_i,\ z_j$ are the parameters entering into the general solution $S$ of the equation $KS = SK$.

6. From items 1, 2, 3, 4, 5 the following follows.

**Theorem 1.** *Let a matrix equation (1) be given, where $A$ and $B$ are two similar integral matrices of order $n$ with elementary divisors over the field of rational numbers*

$$[\varepsilon_i(\lambda)]^{k_{i1}}, \dots, [\varepsilon_i(\lambda)]^{k_{iq_i}} \qquad (i = 1, \dots, t).$$

*The determinant of the matrix $X$—the general solution of equation (1) over the field of rational numbers—decomposes into linear factors if, among each system of exponents $k_{i1}, k_{i2}, \dots, k_{iq}$ $(i = 1, \dots, t)$ of elementary divisors belonging to one and the same irreducible polynomial $\varepsilon_i(\lambda)$, there are no equal ones, and does not decompose into linear factors otherwise.*

7. As one of the applications of the theorem of item 6, let us consider the problem of solving the matrix equation (1) in integral unimodular matrices. This problem divides into two problems:

I. The question of the existence of such solutions.
II. The question of finding such solutions.

Without touching on the first question, let us only note that, for the cases when the characteristic polynomial $\varphi(\lambda)$ of the similar integral matrices $A$ and $B$ is irreducible over the field of rational numbers, it is equivalent to the problem of the equivalence of two ideals of the given field of algebraic numbers (2).

Passing to the second question, let us first suppose that the elementary divisors of the similar integral matrices $A$ and $B$ of order $n$ satisfy the conditions of item 4. To find the required integral unimodular solutions of equation (1) (if such exist), we solve it in integral matrices (4). Then the obtained general solution $X$ is subjected to the condition of unimodularity, i.e., to the condition

$$|X| = \pm 1. \tag{22}$$

It follows from Theorem 1 that equation (22) is either a norm equation or decomposes into a finite number of norm equations and is solved in integers by means of the units of the corresponding fields of algebraic numbers, namely by means of the units of the fields with defining equations $\varepsilon_i(\lambda) = 0$ $(i = 1, 2, \dots, t)$.

If the elementary divisors of the matrices $A$ and $B$ satisfy the conditions of item 5, then the solution of equation (22) in integers reduces, besides the solution of norm equations, to the solution of certain systems of homogeneous Diophantine equations.

8. As another application of Theorem 1, we indicate the following result following from this theorem.

In every cyclic real field of prime degree $p$ there exists a system of conjugate fundamental units if the cyclotomic field $R(\theta)$, $\theta^p = 1$ $(\theta = 1)$, has one ideal class; moreover, there will be infinitely many such systems if $p \geqslant 5$, and a finite number if $p < 5$ ([3], §24).

Dagestan State University
named after V. I. Lenin

Received
13 I 1965

## REFERENCES

[1] A. I. Mal' cev, *Foundations of Linear Algebra*, 1956.
[2] N. G. Chebotarev, *Foundations of Galois Theory*, Part II, 1937.
[3] B. N. Delone, D. K. Faddeev, *Theory of Irrationalities of the Third Degree*, Publishing House of the Academy of Sciences of the USSR, 1940.
[4] B. A. Zeinalov, *Scientific Notes of Dagestan State University named after V. I. Lenin*, **7** (1961).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*