



---

Soviet-era science, translated into English

# E. G. SKLYARENKO

1965

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.31866>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**E. G. SKLYARENKO**

**ON THE DEFINITION OF COHOMOLOGICAL DIMENSION**

*(Presented by Academician P. S. Aleksandrov, 23 X 1964)*

The following definition of the cohomological dimension of a topological space, due to P. S. Aleksandrov <sup>(1)</sup>, is well known.

**Definition 1.** The cohomological dimension  $\dim_G X$  of a space  $X$  with respect to an Abelian group  $G$  is the greatest integer  $n$  for which there exists in  $X$  a closed subset  $A$  such that

$$H^n(X, A; G) \neq 0.$$

In recent papers Okuyama <sup>(3)</sup> used another definition.

**Definition 2.** The cohomological dimension  $D(X; G)$  of a space  $X$  with respect to an Abelian group  $G$  is the least nonnegative integer  $n$  such that, for every set  $A$  closed in  $X$ , the mapping

$$H^m(X; G) \rightarrow H^m(A; G)$$

is epimorphic for all  $m \geq n$ .

An analogous definition was used in <sup>(2)</sup> for locally bicomact spaces.

In the present note the equivalence of these two definitions is proved for paracompact spaces\*.

The inequality  $D(X; G) \leq \dim_G X$  is, obviously, a consequence of the exact cohomology sequence

$$\dots \rightarrow H^m(X, A; G) \rightarrow H^m(X; G) \rightarrow H^m(A; G) \rightarrow H^{m+1}(X, A; G) \rightarrow \dots \quad (*)$$

Let us prove the reverse inequality. Suppose that  $D(X; G) \leq n$ . Since the mapping

$$H^m(X; G) \rightarrow H^m(A; G)$$

is epimorphic for  $m \geq n$ , it follows from the exact sequence (\*) that, for every subset  $A$  closed in  $X$ , the group  $H^{m+1}(X, A; G)$  is isomorphic to a subgroup of the group  $H^{m+1}(X; G)$ . Therefore it is sufficient to prove that the group  $H^{m+1}(X; G)$  is trivial.

For the proof of this we shall need the following lemma.

**Lemma 1.** Let  $h$  be an arbitrary element of the group  $H^{m+1}(X; G)$ . Suppose that the space  $X$  can be represented as the union of a finite number of closed

sets  $X_1, \dots, X_r$ , on each of which the restriction of the element  $h$  is equal to zero. Then  $h = 0$ , if  $D(X; G) \leq m$ .

**Proof** can be carried out by induction on the number of sets  $r$ . For  $r = 1$  the assertion of the lemma is trivial. Suppose that the lemma is true if the number of sets  $X_i$  is  $k - 1$ ; we prove the assertion for  $r = k$ . Let

$$X' = \bigcup_{i=2}^k X_i.$$

Since  $X'$  is closed in  $X$ , we have  $D(X'; G) \leq m$ . Let  $h'$  be the restriction of the element  $h$  to the set  $X'$ . By the induction hypothesis we conclude that  $h' = 0$ . Next consider the exact Mayer-Vietoris cohomology sequence

$$\begin{aligned} \dots \rightarrow H^m(X_1; G) + H^m(X'; G) \xrightarrow{i} H^m(X_1 \cap X'; G) \rightarrow \\ \rightarrow H^{m+1}(X; G) \xrightarrow{j} H^{m+1}(X_1; G) + H^{m+1}(X'; G) \rightarrow \dots \end{aligned}$$

\* Independently, and by another method, this result was also obtained by I. A. Shvedov.

Since  $D(X; G) \leq m$ , the mapping  $i$  is an epimorphism. Consequently,  $\delta = 0$ . Thus the mapping  $j$  is a monomorphism, and since the restriction of the element  $h$  to the sets  $X_1$  and  $X'$  is zero, it follows that  $h = 0$ . The lemma is proved.

Thus, let us prove that  $H^{m+1}(X; G) = 0$ . Let  $h$  be an arbitrary element of the group  $H^{m+1}(X; G)$ . For each point  $x \in X$  there exists a neighborhood  $U_x$  such that the restriction of the element  $h$  to the set  $U_x$  is equal to zero. The sets  $U_x$  form an open covering  $u$  of the space  $X$ . Since the space  $X$  is paracompact, one can inscribe in the covering  $u$  a locally finite covering  $v$  consisting of closed sets  $V_\lambda$ . Moreover, in [4] it is in fact shown that the covering  $v$  can be chosen in such a way that it splits into a countable sequence of subsystems

$$v^i = \{V_\lambda^i\},$$

each of which is discrete (a system of subsets of a space  $X$  is called discrete if at each point of the space  $X$  there exists a neighborhood intersecting no more than one of the sets of the system). Let

$$A_i = \bigcup_{\lambda} V_\lambda^i.$$

Since the covering  $v$  is inscribed in  $u$ , the restriction of the element  $h$  to each of the sets  $V_\lambda^i$  is zero. Since for each  $i$  the set  $A_i$  is the union of a discrete system of closed sets  $V_\lambda^i$ , the restriction of the element  $h$  to each of the closed sets  $A_i$  is also zero. Moreover,

$$X = \bigcup_{i=1}^{\infty} A_i.$$

Consider the following system of sets in the space  $X$ . Since the space  $X$  is paracompact, for the set  $A_1$  there exists a neighborhood  $B_1$  such that the restriction of the cohomology class  $h$  to the closure of this neighborhood

$$C_1 = X[B_1]$$

is zero. Suppose that sets  $B_i, C_i, i \leq k-1$ , have already been constructed, such that each  $B_i$  is open,

$$A_i \cup C_{i-1} \subset B_i, \quad C_i = X[B_i],$$

and the restriction of the element  $h$  to each  $C_i$  is zero. By the lemma, the restriction of the element  $h$  to the set  $C_{k-1} \cup A_k$  is also zero. Therefore, since the space  $X$  is paracompact, for the set  $C_{k-1} \cup A_k$  there exists a neighborhood  $B_k$  such that the restriction of the element  $h$  to the closed set

$$C_k = X[B_k]$$

is zero. Thus one may assume that the sets  $B_i, C_i$  have been constructed for all  $i = 1, 2, \dots$

Since  $A_i \subset B_i \subset C_i$ , we have

$$X = \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} C_i.$$

By construction,

$$B_i \subset C_i$$

and

$$C_i \subset B_{i+1}.$$

Let

$$L_i = C_{i+1} \setminus B_i.$$

The sets  $L_i$  are closed and form a locally finite covering of the space  $X$ , with each  $L_i$  able to intersect only the two elements  $L_{i-1}$  and  $L_{i+1}$  of this covering. Since  $L_i \subset C_{i+1}$ , the restriction of the element  $h$  to each of the sets  $L_i$  is zero. Let, further,  $X_1$  be the union of the sets  $L_i$  with even indices, and  $X_2$  the union of all the remaining ones. The sets  $X_1$  and  $X_2$  are closed; since each of them is the union of a discrete system of sets  $L_i$ , the restriction of the element  $h$  to each of these sets is zero. Finally, since

$$X = X_1 \cup X_2,$$

it follows from the lemma that  $h = 0$ . Since as  $h$  one may take any element of the group  $H^{m+1}(X; G)$ , it follows that

$$H^{m+1}(X; G) = 0.$$

Thus,

$$\dim_G X \leq n.$$

The assertion is proved.

Moscow State University  
named after M. V. Lomonosov

Received  
23 VI 1964

### CITED LITERATURE

1. P. Alexandroff, Proc. Roy. Soc., A, **189**, 11 (1947).
2. H. Cohen, Duke Math. J., **21**, No. 2, 209 (1954).
3. A. Okuyama, Proc. Japan Acad., **38**, No. 8, 489 (1962); **38**, No. 9, 655 (1962).
4. A. H. Stone, Bull. Am. Math. Soc., **54**, 977 (1948).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*