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Abstract

Full Text

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ASYMPTOTICS OF A ONE-DIMENSIONAL SCATTERING PROBLEM

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1. Statement of the problem

Consider the equation

$$y''(x) - \lambda^2 q(x) y(x) = 0, \tag{1}$$

where the function $q(x)$ is real, has a finite number of zeros x_j, $1 \le j \le k$ $(x_j < x_{j+1})$, and

$$-\infty < q(\pm\infty) = q_\pm < 0. \tag{2}$$

Let

$$\int^{\pm\infty} \left|\sqrt{q(x)} - \sqrt{q_\pm}\right| dx < \infty, \qquad \int_{-\infty}^{\infty} |\delta(x)|\, dx < \infty, \tag{3}$$

$$\delta(x) = |{q'}^2(x)| + |q''(x)|.$$

It is known that if conditions (2), (3) are satisfied and $\lambda > 0$, then equation (1) has solutions $y_j^+(x)$ such that, as $x \to +\infty$,

$$y_{1,2}^+(x) \sim |q_+|^{-1/4} \exp\left(\pm i\lambda\sqrt{|q_+|}\, x\right), \tag{4}$$

and the same formulas hold for $y_j^-(x)$ as $x \to -\infty$. Then

$$\begin{pmatrix} y_1^+ \\ y_2^+ \end{pmatrix} = S(\lambda) \begin{pmatrix} y_1^- \\ y_2^- \end{pmatrix}; \tag{5}$$

$S(\lambda)$ is a matrix of order two, with $s_{22} = \bar{s}_{11}$, $s_{21} = \bar{s}_{12}$. The quantities

$$D_+ = \sqrt{\frac{q_+}{q_-}}\,|s_{11}|^{-2}, \qquad R_+ = |s_{21}s_{11}^{-1}|^2 \tag{6}$$

are called the transmission and reflection coefficients, respectively, and $D_+ + R_+ = 1$.

Our aim is to find the asymptotics of $S(\lambda)$ and $D_+(\lambda)$ as $\lambda \to +\infty$. We shall consider this problem under the assumption that $q(z)$ is an entire function of z, having only simple zeros. On $q(z)$ we impose such conditions that make it possible to construct solutions of equation (1) admitting asymptotic expansions in series in powers of λ^{-1}. Because of the unwieldiness of these conditions, we shall not write them down here. Conditions on $q(z)$ that make it possible to obtain the first terms of the asymptotic expansion are given in [1].

2. The problem of transmission through a barrier

In this case $q(x)$ has no zeros. Introduce the notation

$$\xi(z_1, z_2) = \int_{z_1}^{z_2} \sqrt{q(z)}\,dz; \tag{7}$$

$$\alpha_j = \left|\xi(x_{2j}, x_{2j+1})\right|, \qquad c_j = \xi(x_{2j-1}, x_{2j}) > 0; \tag{8}$$

$$C_- = x_1\sqrt{|q_-|} + \int_{-\infty}^{x_1} \left(\sqrt{|q(x)|} - \sqrt{|q_-|}\right) dx, \qquad A_- = e^{i\lambda C_-}; \tag{9}$$

$$C_+ = -x_k\sqrt{|q_+|} + \int_{x_k}^{+\infty} \left(\sqrt{|q(x)|} - \sqrt{|q_+|}\right) dx, \qquad A_+ = e^{i\lambda C_+}. \tag{10}$$

Theorem 1. Let $q(x)$ have $2m$ zeros and satisfy the conditions formulated above. Then, as $\lambda \to +\infty$, $m > 1$,

$$s_{11} = -i(A_+A_-)^{-1}s_{11}^0, \qquad s_{21} = A_-A_+^{-1}s_{21}^0; \tag{11}$$

$$s_{11}^0 = 2^{m-i}\exp\left(\lambda\sum_{j=1}^{m} c_j\right)\left(\prod_{j=1}^{m-1}\cos\lambda\alpha_j + O(\lambda^{-1})\right); \tag{12}$$

s_{21}^0 has the same form as s_{11}^0. For $m = 1$, the last factor in (12) should be replaced by $1 + O(\lambda^{-1})$.

For $m = 1, 2$ and in some special cases with $m > 2$, these formulas were obtained earlier (see ([2–6])). We note that, for the proof of formulas (11), (12), it is enough to impose on $q(x)$ conditions (2), (3).

As follows from Theorem 1, the coefficient $D_+(\lambda)$ has, for $m > 1$, maxima near the points

$$\lambda_{nj}^0 = \pi\alpha_j^{-1}(n+1/2), \qquad 1 \le j \le m-1.$$

The values of λ for which $D_+(\lambda)$ has maxima are called **resonance** values. We shall study the following case in more detail.

3. The case $m = 2$.

Theorem 2. Let the conditions of Theorem 1 be satisfied, $m = 2$. Then, for sufficiently large $\lambda > 0$, the maxima of $D_+(\lambda)$ are attained at the points λ_n, $n_0 \le n < \infty$, and

$$1^\circ. \quad D_+(\lambda_n) \sim 4\sqrt{\frac{q_+}{q_-}}\exp[-2\lambda_n|c_1 - c_2|], \tag{13}$$

if $c_1 \ne c_2$,

$$2^\circ. \quad D_+(\lambda_n) = 1 + O(n^{-2}), \tag{14}$$

if $c_1 = c_2$. If $q(x)$ is an even function, then $D_+(\lambda_n) = 1$.

Formula (13) is unknown even in the physical literature.

For λ_n there is an asymptotic expansion as $n \to \infty$

$$2\lambda_n\alpha_1 \sim 2n\pi + \pi - \sum_{k=1}^{\infty}(\lambda_n)^{-k}\int_C \alpha_k(z)\,dz, \tag{15}$$

where C is a closed contour enclosing the segment $[x_2, x_3]$ and not containing inside itself other zeros of $q(z)$. For the functions $\alpha_k(z)$, see ([1]). Analogous expansions hold for λ_{nj} $(m > 2)$.

Theorem 3. Let the conditions of Theorem 1 be satisfied, $c_1 \ne c_2$. Then, for $n > n_0$, there exist complex values λ_n^* such that $D_+(\lambda_n^*) = 1$ and

$$\lambda_n^* - \lambda_n \sim \frac{i}{4}\alpha_1^{-1}\exp(-2\lambda_n c_0)\,\varepsilon, \tag{16}$$

$$c_0 = \min(c_1, c_2), \qquad \varepsilon = \operatorname{sign}(c_1 - c_2).$$

4. The overbarrier reflection problem. In this case $q(x)$ has no zeros and there exists a domain $D \supset Ox$ such that $\xi(D)$ is the strip $-a < \operatorname{Re}\xi < a$ ($\xi = \xi(0,x)$), on Γ—the boundary of D—lie zeros $q(z)$. The domain D is symmetric with respect to Ox. Let Γ^+ be the part of Γ lying in the upper half-plane. The case when Γ^+ contains exactly 1 zero was investigated by physicists in ([7],[8]).

Theorem 4. Let $q(x)$ have no zeros, satisfy the conditions formulated in item 1, and let Γ^+ contain exactly two zeros z_1, z_2 of the function $q(z)$. Then, as $\lambda \to +\infty$,

$$s_{12} = -2i \exp\{-\lambda(c_1 + c_2)\} \left(\cos \lambda\alpha + O(\lambda^{-1})\right) s_{12}^0, \tag{17}$$

$$s_{12}^0 = (A_+^0 A_-^0)^{-1}, \qquad c_j = \xi(0, z_j), \qquad \alpha = |\xi(z_1, z_2)|.$$

Here $\operatorname{Re} c_j > 0$, and $A_\pm^0$ are determined by formulas (9), (10), where $x_1 = x_k = 0$.

In this case, near $\lambda_n = \pi\alpha^{-1}(n + 1/2)$ lie resonance values of λ.

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REFERENCES

1. M. V. Fedoryuk, DAN, 158, No. 3 (1964).

2. D. Bohm, *Quantum Theory*, 1961.

3. I. I. Gol' dman, V. D. Krivchenkov, *Collection of Problems in Quantum Mechanics*, 1957.

4. V. P. Maslov, DAN, 151, No. 2 (1963).

5. V. A. Pshenichnov, *Solid State Physics*, 4, issue 5 (1962).

6. L. D. Landau, E. M. Lifshitz, *Quantum Mechanics*, 1964.

7. V. L. Pokrovskii, S. K. Savvinykh, F. R. Ulinich, ZhETF, 34, issue 7 (1958).

8. V. L. Pokrovskii, I. M. Khalatnikov, ZhETF, 40, issue 6 (1961).

Note: Figure translations are in progress. See original paper for figures.