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Abstract

Full Text

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CONTINUOUS DEPENDENCE OF SOLUTIONS OF A SINGULAR INTEGRAL EQUATION WITH SHIFT

(Presented by Academician I. N. Vekua, January 18, 1965)

In the present paper we study the solution of the singular integral equation

$$u(x) = \int_a^b \frac{K[x, s, u(s)]}{s - \alpha(x)}\, ds. \tag{1}$$

The function $y = \alpha(x)$ maps the segment $[a, b]$ one-to-one onto itself with preservation of orientation; moreover,

$$0 < m \leq \left| \frac{\alpha(x_2) - \alpha(x_1)}{x_2 - x_1} \right| \leq M < \infty \tag{2}$$

for all $a \leq x_1, x_2 \leq b$. Hence $\alpha(x)$ is a monotone function, $\alpha(a) = a$, $\alpha(b) = b$, and there exists an inverse function $x = \beta(y)$ to $\alpha(x)$, mapping $[a, b]$ onto itself with preservation of orientation and

$$0 < \frac{1}{M} \leq \left| \frac{\beta(y_2) - \beta(y_1)}{y_2 - y_1} \right| \leq \frac{1}{m} < \infty. \tag{2^1}$$

The function $K(x, s, u)$ $(a \leq x, s \leq b,\ |u| \leq N)$ satisfies the conditions

$$|K(x + \Delta x, s + \Delta s, u + \Delta u) - K(x, s, u)| \leq \tag{1}$$

$$\leq N_1 \psi(|\Delta x|) + N_2 \varphi(|\Delta s|) + N_3 |\Delta u|; \tag{3}$$

$$K(a, a, u) \equiv K(b, b, u) \equiv 0; \tag{4}$$

$$|g(x, s, u) - g(x, s, v)| \leq N_4 \varphi_1(|x - s|)|u - v|, \tag{5}$$

where $g(x,s,u) = K(x,s,u) - K(s,s,u)$; the functions $\psi(t)$, $\varphi(t)$, $\varphi_1(t)$ belong to the space Φ for $0 < t \leq b - a$. (For the space Φ and the classes $H_n(\varphi)$, see works ([1,3]).) The author has proved the existence and uniqueness of a solution of equation (1) in the class $H_n(\varphi)$, provided the functions $\alpha(x)$ and $K(x,s,u)$ satisfy conditions (2), (3), (4), (5).

Theorem. *If $K(x,s,u)$ and $\alpha(x)$ satisfy conditions (2), (3), (4), (5), then the solution of equation (1), for sufficiently small $|\lambda|$, depends continuously on the function $\alpha(x)$.*

We give a scheme of the proof of the theorem. Let $u_2(x)$ and $u_1(x)$ be solutions of equations (1) corresponding to $\alpha_2(x)$ and $\alpha_1(x)$, and let

$$\max_{a \leq x \leq b} |\alpha_2(x) - \alpha_1(x)| < \varepsilon.$$

Set $\alpha_1(x) = y$ $(a \leq x \leq b)$. Then $x = \beta(y)$ $(a \leq y \leq b)$;

$$\alpha_2(x) = \alpha_1(x) + [\alpha_2(x) - \alpha_1(x)] = y + \gamma(y),$$

$$\gamma(a) = \gamma(b) = 0, \qquad |\gamma(y)| < \varepsilon \quad (a \leq y \leq b).$$

The function

$$K^*(y,s,u) = K[\beta(y), s, u]$$

satisfies conditions (3), (4), (5), only N_1 and N_4 change.

We shall have

$$u_2[\beta(y)] - u_1[\beta(y)] = \lambda \int_a^b \frac{K^*[y,s,u_2(s)]}{s - y - \gamma(y)}\, ds - \lambda \int_a^b \frac{K^*[y,s,u_1(s)]}{s-y}\, ds =$$

$$= \lambda(Bu_2 - Bu_1) + \lambda I^*, \tag{6}$$

where

$$Bu = \int_a^b \frac{K^*[y,s,u(s)]}{s-y}\, ds,$$

$$I^* = \int_a^b \frac{K^*[y+\gamma(y), s, u_2(s)]}{s - y - \gamma(y)}\, ds - \int_a^b \frac{K[y,s,u_2(s)]}{s-y}\, ds +$$

$$+\int_a^b \frac{K^*[y,s,u_2(s)] - K^*[y+\gamma(y),s,u_2(s)]}{s-y-\gamma(y)}\,ds = \widetilde{B}u_2 - Bu_1 + I. \qquad (7)$$

If we make the substitution $y = \alpha(x)$, we obtain

$$I = \int_a^b \frac{f(x,s) - f[x,\alpha_2(x)]}{s-\alpha_2(x)}\,ds + f[x,\alpha_2(x)]\int_a^b \frac{ds}{s-\alpha_2(x)} = I_1 + I_2,$$

where $f(x,s) = K^*[\alpha_1(x),s,u_2(s)] - K^*[\alpha_2(x),s,u_2(s)]$.

The function $f(x,s)$ satisfies the conditions

$$|f(x+\Delta x, s+\Delta s) - f(x,s)| \le N_5\{\psi(|\Delta x|) + \varphi(|\Delta s|)\}; \qquad (3')$$

$$f(a,a) = f(b,b) = 0; \qquad (4')$$

$$|f(x,s)| \le N_1'\psi[|\alpha_2(x) - \alpha_1(x)|] \le N_1'\psi(\varepsilon) \qquad (8)$$

for $a \le x,\ x+\Delta x,\ s,\ s+\Delta s \le b$.

Using (8), $(3')$, and the properties of the functions of the space Φ, we obtain

$$|I_1| \le \{2N_1'\psi(\varepsilon)\}^{1-\mu} N_5^{\mu} \int_a^b \frac{[\varphi(|s-\alpha_2(x)|)]^{\mu}}{|s-\alpha_2(x)|}\,ds \le M_1[\psi(\varepsilon)]^{1-\mu},$$

$$|I_2| \le M_2[\psi(\varepsilon)]^{1-\mu},$$

$$|I| \le M_3[\psi(\varepsilon)]^{1-\mu},$$

where $0 < \mu < 1$ is arbitrary.

In proving the existence of a solution of equation (1), the estimate obtained is

$$|\widetilde{B}u_2 - Bu_1| \le M_4[\gamma(y)] \le M_4\varphi(\varepsilon).$$

Combining the last two inequalities, according to (7), we have

$$|I^*| \le M_5[\varphi(\varepsilon)]^{1-\mu}.$$

According to (6), we have

$$|u_2(\beta(y)) - u_1[\beta(y)]| \leq |\lambda|\,|Bu_2 - Bu_1| + |\lambda| M_5[\varphi(\varepsilon)]^{1-\mu}.$$

Hence it follows that

$$\|u_2 - u_1\|_{L_p} \leq |\lambda|\,\|Bu_2 - Bu_1\|_{L_p} + |\lambda| M_5[\varphi(\varepsilon)]^{1-\mu}(b-a)^{1/p}.$$

In the proof of the uniqueness theorem for the solution of equation (1) (analogously to (2)), one obtains the inequality

$$\|Bu_2 - Bu_1\|_{L_p} \leqslant c\,\|u_2 - u_1\|_{L_p},$$

where $c > 0$ is some constant.

If λ is chosen so that $|\lambda|c < 1$ (this is the condition for uniqueness of the solution of equation (1)), we obtain

$$\|u_2 - u_1\|_{L_p} \leqslant M_6[\varphi(\varepsilon)]^{1-\mu}. \tag{9}$$

Kh. Sh. Mukhtarov has established that for any $u(x) \in H_N(\varphi)$ one has

$$\max_{a\leqslant x\leqslant b} |u(x)| \leqslant DN^{\alpha/p}\left\{\|u\|_{L_p}\right\}^{(p-\alpha)/p^2}, \tag{10}$$

where $p \geqslant 1$ is arbitrary, and $0 < \alpha < p$ is a certain definite number; D is a constant independent of N.

If (10) is applied to inequality (9), we finally obtain

$$\max_{a\leqslant x\leqslant b} |u_2(x) - u_1(x)| \leqslant M_7[\varphi(\varepsilon)]^{(p-\alpha)(1-\mu)/p^2},$$

where M_7 is a constant depending only on $N_1, N_2, N_3, N_0, \varphi(t), \varphi_1(t), m, M, (b-a)$.

In conclusion we report that, for equation (1), the author has proved the continuous dependence of the solutions on $K(x,s,u)$ (stability of solutions), and also that the solution of equation (1) with respect to the parameter λ (for small λ) belongs to the Hölder class.

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Note: Figure translations are in progress. See original paper for figures.