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Abstract

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V. V. ZHIKOV

ON ABSTRACT EQUATIONS WITH ALMOST-PERIODIC COEFFICIENTS

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The paper defines abstract N-almost-periodic functions, formulates for them an approximation theorem and a uniqueness theorem, and considers the question of finding almost-periodic and N-almost-periodic solutions of abstract equations with almost-periodic coefficients.

I. Let B be a Banach space and let $f(x)$ be a continuous function of the real variable x with values in B.

Definition 1. A number $\tau = \tau(\varepsilon, N)$ is called an ε, N-almost-period if, for all $|x| \le N$, the inequality

$$\|f(x+\tau) - f(x)\| \le \varepsilon$$

holds.

Definition 2. A continuous function $f(x)$ is called an N-almost-periodic function if one can specify a countable sequence of numbers $\Lambda_1, \Lambda_2, \dots, \Lambda_n, \dots$ having the following property: whatever positive numbers ε, N may be, one can specify an integer $n = n(\varepsilon, N)$ and a positive number $\delta(\varepsilon, N)$ such that every real number τ satisfying the system of inequalities

$$|\Lambda_k \tau| \le \delta \pmod{2\pi} \qquad (k = 1, \dots, n)$$

is an ε, N-almost-period of $f(x)$.

It is known ([1]) that with every ordinary (numerical) N-almost-periodic function there can be associated a certain set of Fourier series; moreover, an N-almost-periodic function is uniquely determined by any one of its Fourier series. It turns out that an analogous circumstance also holds for abstract N-almost-periodic functions under sufficiently general assumptions.

Let $f(x)$ be an N-almost-periodic function such that

$$\overline{\lim} \frac{1}{2T} \int_{-T}^{T} \|f(x)\| \, dx < C. \tag{1}$$

Choose $\Lambda_1, \dots, \Lambda_n, \dots$ according to Definition 2, and denote by $M_1, \dots, M_n, \dots$ the least module containing the numbers $\Lambda_1, \dots, \Lambda_n, \dots$. Take an arbitrary sequence of positive numbers $T_m \to \infty$ and consider the sequence

$$\frac{1}{2T_m} \int_{-T_m}^{T_m} f(x) e^{iM_{nx}}\, dx.$$

In view of (1), this sequence is bounded and therefore has at least one weak limit point in B^{**}. For each n, choose some one limit point. The expression

$$\widetilde{\lim}_{m\to\infty} \frac{1}{2T_m} \int_{-T_m}^{T_m} f(x) e^{iM_n}\, dx$$

will denote precisely this fixed Fourier–Bohr coefficient. Let us introduce the polynomials

$$P_{np}(x) = \frac{1}{K_{np}} \widetilde{\lim}_{m\to\infty} \frac{1}{2T_m} \int_{-T_m-x}^{T_m-x} f(x+t) S_n^{2p}(t)\, dt,$$

where

$$S_n = \frac{1}{n}(\cos\Lambda_1 t + \cos\Lambda_2 t + \dots + \cos\Lambda_n t) + \alpha; \qquad k_{np} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} S_n^{2p}\, dt,$$

with α depending only on $\{\Lambda_1, \dots, \Lambda_n\}$.

Theorem 1. *Whatever the positive numbers ε, N may be, one can indicate integers n, p such that for all $|x| \le N$ the inequality*

$$\|f(x) - P_{np}(x)\| \le \varepsilon$$

holds.

Corollary. From Theorem 1 it follows that if two N-almost-periodic functions have the same Fourier series, then, as is not difficult to see, they are approximated on any finite interval by the same trigonometric sums and, consequently, coincide. Below we shall see that N-almost-periodic functions arise in the solution of abstract equations with almost-periodic coefficients (for ordinary equations see (1), Ch. IV).

II. Let B be a Banach space, uniformly convex, separable, with separable conjugate. Consider the following nonstationary problem:

$$x'(t) = A(t)x(t) + f(t), \qquad x(0) = x_0. \tag{S}$$

It is assumed that the domains of definition of the operator $A(t)$ contain everywhere a dense set M_0, independent of t, while the domains of definition of the adjoint operators $A^*(t)$ contain everywhere a dense set L_0. Problem (S) is solvable for every $x_0 \in B$, and there is continuous dependence on the initial data. It is also assumed that $f(t)$, $A(t)x$, and $A^*(t)y$ are almost-periodic functions for every $x \in M_0$ and $y \in L_0$. Hence it follows that for any sequence of real numbers h_n there exists a subsequence h'_n and an operator function $B(t)$ such that for any $y \in B^*$ and $x \in M_0$, uniformly in t,

$$\langle yA(t+h'_n)x\rangle \to \langle yB(t)x\rangle. \tag{2}$$

In view of (2), we shall write

$$\lim_{n\to\infty} A(t+h'_n) = B(t). \tag{3}$$

Relation (3) shows that limiting operator functions are defined. The Cauchy problem for limiting operator functions and limiting free terms, analogous to problem (S), will be denoted by (Σ_h).

Let us assume further that if $\lim_{n\to\infty} A(t+h_n) = B(t)$, $\lim_{n\to\infty} f(t+h_n) = g(t)$, and

$$x'(t+h_n) = A(t+h_n)x(t+h_n) + f(t+h_n), \qquad z(t) = B(t)z + g(t),$$

$x_n(0) \to z(0)$, then $x_n(t) \to z(t)$ for every t.

Let Ω be the set of bounded solutions of the nonhomogeneous problem (S). Put $\mu(x) = \sup_{-\infty<t<\infty} \|x(t)\|$ for $x(t) \in \Omega$. Let $\mu_{Af} = \inf_{x(t)\in\Omega} \mu(x)$. If there exists $x^0(t) \in \Omega$ such that $\mu(x^0) = \mu_{Af}$, then one says that a minimal solution exists.

Under the assumptions indicated above, the following theorems hold:

Theorem 2. *If the nonhomogeneous problem (S) has a bounded solution, while the homogeneous problem has no bounded solutions whose norms can be arbitrarily small (apart from the trivial one), then the minimal solution exists and is unique. If, moreover, the minimal solution is not a solution of any of the limiting problems distinct from problem (S), then it is weakly N-almost-periodic.*

Theorem 3. *If the nonhomogeneous problem (S) has a bounded solution, while the homogeneous problems (Σ_h) have no nontrivial bounded solutions whose norms can be arbitrarily small, then the minimal solution is weakly almost-periodic.*

Let us note a special case, when $A(t) = A_0K(t)$, where $K(t)$ is an operator-valued function, almost-periodic in the operator norm, and A_0 is a certain, in general unbounded, operator independent of time.

Theorem 4. *Under the same conditions on bounded solutions of the homogeneous problem* (S) *as in Theorem 2, the minimal solution is weakly* N*-almost-periodic.*

To establish strong N-almost-periodicity, additional assumptions are needed. Namely, if $\lim_{n\to\infty} A(t+h_n) = B(t)$, $\lim_{n\to\infty} f(t+h_n) = g(t)$, $x_n'(t+h_n) = A(t+h_n)x(t+h_n) + f(t+h_n)$, $z_n'(t) = B(t)z_n(t) + g(t)$, $x_n(0) = z_n(0)$, and $\|x_n(0)\| < C$, then $\|x_n(t) - z_n(t)\| \to 0$ for every t.

Moreover, for every $M > 0$ one can indicate $K_M > 0$ such that

$$\inf_{-\infty<t<\infty} \|u(t)\| \geq K_M \sup_{-\infty<t<\infty} \|u(t)\| \tag{4}$$

for every bounded solution of the homogeneous problem (S) such that $\|u(t)\| \leq M$. Under these conditions the following holds.

Theorem 5. *A minimal weakly* N*-almost-periodic solution is strongly* N*-almost-periodic.*

Theorem 6. *If condition (4) is fulfilled for all limiting problems* (Σ_n)*, then the minimal solution is almost-periodic.*

Let us note that Theorems 3 and 6, under somewhat different assumptions, were indicated by Amerio ([2]). We proceed to consider some examples.

III. Consider the problem

$$u'' = -(A^2 + iB)u + f(t), \qquad u(0) = u_0, \qquad u'(0) = u_1, \tag{5}$$

where A is a positive self-adjoint operator independent of time; $f(t)$ is an almost-periodic function. In addition, the operator $A^2 + iB$ has a complete system of eigenfunctions and associated functions, and B is completely continuous relative to A. In the product $H \times H$ introduce the norm by the formula $\|v\|^2 = \|\{x, y\}\|^2 = (x, x) + (Ay, Ay)$, and denote the Hilbert space thus obtained by W_A. It is known ([3]) that problem (5) is solvable for every $u_0 \in D(A)$ and $u_1 \in H$, and the solution $v(t) = \{u'(t), u(t)\}$ is a continuous function in W_A.

Theorem 7. *Every* W_A*-bounded solution of problem (5) is* W_A*-almost-periodic.*

We briefly outline the proof. First of all, weak W_A-almost-periodicity is a consequence of the completeness of the eigenfunctions and associated functions and of the classical Favard theorems.

Let $\|v(t)\|^2_{W_A} = (u', u') + (Au, Au) = E(t)$. Differentiating this expression, we obtain, by virtue of (5),

$$E'(t) = (f, u') + (u', f) + i(Bu, u') - i(Bu, u'),$$

whence

$$E(t) = E(0) + \int_0^t (f, u') + (u', f) + i(Bu, u') - i(Bu, u'). \qquad (6)$$

The calculations carried out are valid for solutions satisfying (5) in the strong sense. However, formula (6) extends by continuity to solutions lying in W_A.

Substitute into (6) a bounded solution in W_A. It is not difficult to show that the integrand in (6) is an almost-periodic function (in the sense of Bohr); hence, by the Bohr theorem, $E(t) = \|v(t)\|^2_{W_A}$ is an almost-periodic function. The same reasoning may also be applied to a solution $\hat{v}(t)$ that is a limiting translate (in the weak sense) of the solution $v(t)$, i.e., we obtain that $\|\hat{v}(t)\|^2_{W_A}$ is almost periodic. Hence, by one lemma of Amerio [4], it is not difficult to conclude that $v(t)$ is a W_A-almost-periodic function. This completes the proof of Theorem 7.

We note that the first results concerning the almost periodicity of solutions of abstract equations of general form were obtained by Bochner [5]. His results are, however, conditional in nature. Using them, S. L. Sobolev proved the almost periodicity of the homogeneous solution of the wave equation [6]. The final results concerning the general inhomogeneous wave equation are due to Amerio. We note that in Amerio' s works the existence of a constant energy integral is essentially used, which is possible only in the self-adjoint case.

A theorem analogous to Theorem 7 can also be formulated in the case when $A(t)$ depends on time, namely, $A(t)$ is periodic and $A(t_1)$ and $A(t_2)$ commute for arbitrary t_1 and t_2.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.