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Abstract

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MATHEMATICS

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ON DECOMPOSITIONS OF INFINITELY DIVISIBLE LAWS WITHOUT A GAUSSIAN COMPONENT

(Presented by Academician Yu. V. Linnik, 30 IX 1964)

1°. Introduction. We shall use the terminology adopted in the monograph [1]. In particular, by the Poisson spectrum of an infinitely divisible (i.d.) law F we shall mean the set of growth points of the functions $M_1(x)$ and $M_2(x)$ occurring in the representation of its characteristic function (c.f.) $\varphi(t)$ by Lévy' s formula

$$\varphi(t) = \exp\left\{ i\alpha t - \gamma t^2 + \int_{-\infty}^{0} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dM_1(x) + \right.$$

$$\left. + \int_{0}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dM_2(x) \right\}.$$

By I_0 we shall denote the class of i.d. laws having only i.d. components.

Yu. V. Linnik proved ([1], Ch. VIII) that, for an i.d. law with a Gaussian component to belong to I_0, it is necessary that the Poisson spectrum of this law be finite or countable and, in addition, satisfy a very stringent condition of arithmetic character. We shall consider i.d. laws without a Gaussian component. From the theorems established by P. Lévy [2] and D. A. Raikov [3] it follows that, for such laws, the condition of arithmetic character appearing in Yu. V. Linnik' s result is not necessary for membership in I_0. However, it was not known ([1], p. 257) whether it is necessary that the Poisson spectrum be finite or countable.

We give a negative answer to this question, indicating two classes of i.d. laws lying in I_0 such that among the components of each of them there are laws with continuous Poisson spectrum.

2°. Formulation of the results.

Theorem 1. *Let F be an i.d. law without a Gaussian component, whose Poisson spectrum lies on the segment $[a, b]$, and suppose that the condition $0 < a < b \leq 2a < \infty$ is fulfilled. Then $F \in I_0$.*

Following [4], we shall call a set with linearly independent points a set on the real axis such that every finite system of its points is linearly independent over the field of rational numbers. It is known ([4], p. 209) that every perfect set contains a perfect subset with linearly independent points.

Theorem 2. *Let F be an i.d. law without a Gaussian component, whose Poisson spectrum is positive and is a closed bounded set with linearly independent points. Then $F \in I_0$.*

Theorems 1 and 2 describe the two classes of i.d. laws lying in I_0 which were mentioned in the introduction. We derive Theorems 1 and 2 from a single theorem, possibly of independent interest, on the form of the c.f. of components of an i.d. law without a Gaussian component and with positive closed bounded Poisson spectrum. To formulate it, we introduce the following notation: $V^{(b)}$ is the collection of all funct-

tions having bounded variation on the axis $(-\infty, \infty)$, continuous from the left and equal to zero at $-\infty$; $S(\sigma)$, where $\sigma \in V^{(b)}$, is the smallest closed set whose complement consists of points of constancy of $\sigma(x)$; $(n)A$, where $n = 1, 2, 3, ...,$ is the set defined by the conditions: $A = A$, $(n)A = (n-1)A + A$ ($A + B$ is the set consisting of numbers of the form $\alpha + \beta$, where $\alpha \in A$, $\beta \in B$).

Theorem 3. Let F be an i.d. law without a Gaussian component, with Poisson spectrum A, and let $0 < a < b < \infty$, where $a = \inf_{x \in A} x$, $b = \sup_{x \in A} x$. Then the ch.f. of any component of the law F has the form

$$\exp\left\{i\alpha t + \int_{-\infty}^{\infty} (e^{ixt} - 1)\, d\sigma(x)\right\}, \tag{1}$$

where α is real, and the function $\sigma(x) \in V^{(b)}$ is nondecreasing on $[a, 2a]$ and

$$S(\sigma) \subset [a, b] \cap \bigcup_{n=1}^{\infty} (n)A.$$

Theorems 1, 2, and 3 for the case when the Poisson spectrum of the law F consists of a finite number of points were proved by D. A. Raikov (^{3}). The method that we use in this article is a certain development of D. A. Raikov' s method.

3°. Results used.

(a) If $\sigma_1(x), \sigma_2(x) \in V^{(b)}$, then $S(\sigma_1 * \sigma_2) = \overline{S(\sigma_1) + S(\sigma_2)}$; moreover, if $\sigma_1(x)$ and $\sigma_2(x)$ are nondecreasing functions, then $S(\sigma_1 * \sigma_2) = S(\sigma_1) + S(\sigma_2)$.

(b) (A special case of Theorem 9.02 from (^{1}), p. 184). If F is an i.d. law whose Poisson spectrum lies in $[-b, b]$, $b > 0$, then the ch.f. of any component of the law F has the form $\exp g(t)$, where $g(t)$ is an entire function of exponential type not exceeding b.

(c) (Corollary of the Paley–Wiener theorem). If an entire function $g(t)$ of exponential type not exceeding b is representable for real t in the form

$$g(t) = \int_{-\infty}^{\infty} e^{ixt}\, d\sigma(x), \tag{2}$$

where $\sigma(x) \in V^{(b)}$, then $S(\sigma) \subset [-b, b]$, and representation (2) holds in the whole complex t-plane.

4°. Proof of Theorem 3. Without loss of generality, one may assume that the ch.f. of the law F has the form

$$\varphi_0(t) = \exp\left\{\int_{-\infty}^{\infty} (e^{ixt} - 1)\, d\sigma_0(x)\right\},$$

where $\sigma_0(x) \in V^{(b)}$ is a nondecreasing function and $S(\sigma_0) = A$. Then we have

$$\varphi_0(t) = c_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left\{\int_{-\infty}^{\infty} e^{ixt}\, d\sigma_0(x)\right\}^n,$$

where $c_0 = \exp\{\sigma_0(-\infty) - \sigma_0(\infty)\}$. Hence the relation* follows:

$$F(x) = c_0 \left\{\varepsilon(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \sigma_0^{n*}(x)\right\}, \tag{3}$$

where $\varepsilon(x) = 0$ for $x \leq 0$, $\varepsilon(x) = 1$ for $x > 0$. By virtue of (a) we have $S(\sigma_0^{n*}) = (n)S(\sigma_0) = (n)A$. Since $(n)A \subset [na, nb]$, $a > 0$, for each can intersect a finite interval for only a finite number of sets $(n)A$. Therefore the set $\bigcup_{n=1}^{\infty} (n)A$ is closed, and from (3) it follows that

$$S(F) = \{0\} \cup \bigcup_{n=1}^{\infty} (n)A.$$

Now let $F = F_1 * F_2$, where F_1 and F_2 are some probability laws. Since (by (a)) $S(F) = \overline{S(F_1) + S(F_2)}$, and $S(F) \subset [0, \infty]$, the sets $S(F_1)$ and $S(F_2)$ are bounded on the left. Without loss of generality, we may assume that the infimum

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* If $\sigma \in V^{(b)}$, then σ^{n*} is defined by the conditions $\sigma^{1*} = \sigma$, $\sigma^{n*} = \sigma * \sigma^{(n-1)*}$.

for $S(F_1)$ is 0 (this can be achieved by replacing $F_1(x)$ by $F_1(x+\delta)$, and $F_2(x)$ by $F_2(x-\delta)$, where δ is the infimum of the set $S(F_1)$). But then $0 \in S(F_1)$, $0 \in S(F_2)$, and, consequently, $S(F_1) \cup S(F_2) \subset S(F)$.

Since 0 is an isolated point for $S(F)$, 0 is also an isolated point for $S(F_1)$. Therefore the law F_1 has a jump at the point 0. Denoting the size of the jump by c_1 ($c_1 > 0$), we may write the relation: $F_1(x) = c_1\varepsilon(x) + G(x)$, in which $G(x) \in V^{(b)}$ is nondecreasing and

$$S(G) \subset \bigcup_{n=1}^{\infty} (n)A \subset [a, \infty).$$

For the c.f. $\varphi_1(t)$ of the law F_1, we obtain the expression

$$\varphi_1(t) = c_1 + \int_a^{\infty} e^{ixt}\, dG(x). \tag{4}$$

By virtue of (b) we have $\varphi_1(t) = \exp g(t)$, where $g(t)$ is an entire function of exponential type not exceeding b, and, consequently, the function $\varphi_1(t)$ is entire. Then relation (4) holds in the whole complex t-plane.

Choose $\eta > 0$ so large that

$$\int_a^{\infty} e^{-x\eta}\, dG(x) < c_1.$$

Put

$$G_\eta(x) = \int_{-\infty}^{x} e^{-y\eta}\, dG(y).$$

Noting that for real ξ

$$\left| \int_a^{\infty} e^{ix\xi}\, dG_\eta(x) \right| \le \int_a^{\infty} dG_\eta(x) < c_1, \tag{5}$$

we obtain the relation

$$g(\xi + i\eta) = \ln \varphi_1(\xi + i\eta) = \ln \left\{ c_1 + \int_a^{\infty} e^{ix\xi}\, dG_\eta(x) \right\}$$

$$= \ln c_1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n c_1^n} \left\{ \int_a^{\infty} e^{ix\xi}\, dG_\eta(x) \right\}^n \tag{6}$$

(ξ real). Consider now the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{nc_1^n}\, G_\eta^{n*}(x).$$

From the right-hand inequality in (5) it follows easily that this series converges on $(-\infty, \infty)$ and that its sum belongs to $V^{(b)}$. Denote this sum by $\sigma_\eta(x)$. Since $S(G_\eta) = S(G)$, we have

$$S(\sigma_\eta) \subset \bigcup_{n=1}^{\infty}(n)S(G) \subset \bigcup_{n=1}^{\infty}(n)\left\{\bigcup_{k=1}^{\infty}(k)A\right\} = \bigcup_{n=1}^{\infty}(n)A.$$

Further, from (6) it follows that for real ξ

$$g(\xi + i\eta) = \ln c_1 + \int_{-\infty}^{\infty} e^{ix\xi}\, d\sigma_\eta(x). \tag{7}$$

With the aid of (b) we conclude that $S(\sigma_\eta) \subset [-b, b]$ and that relation (7) holds for all complex ξ. Setting in (7) $\xi = t - i\eta$ and taking

$$\sigma(x) = \int_{-\infty}^{x} e^{y\eta}\, d\sigma_\eta(y),$$

we shall have

$$g(t) = \ln c_1 + \int_{-\infty}^{\infty} e^{ixt}\, d\sigma(x), \tag{8}$$

where

$$S(\sigma) = S(\sigma_n) \subset [-b, b] \cap \bigcup_{n=1}^{\infty}(n)A = [a, b] \cap \bigcup_{n=1}^{\infty}(n)A.$$

Now we have

$$\varphi_1(t) = \exp g(t) = c_1 \sum_{n=0}^{\infty} \frac{1}{n!}\left\{\int_{-\infty}^{\infty} e^{ixt}\, d\sigma(x)\right\}^n.$$

Hence we obtain

$$F_1(x) = c_1\left\{\varepsilon(x) + \sum_{n=1}^{\infty} \frac{1}{n!}\sigma^{n*}(x)\right\}. \tag{9}$$

Since

$$S\left(\sum_{n=2}^{\infty}\frac{1}{n!}\sigma^{n*}\right)\subset\bigcup_{n=2}^{\infty}(n)S(\sigma)\subset[2a,\infty),$$

it follows from (9), for $a\leq x_1<x_2\leq 2a$, that

$$F_1(x_2)-F_1(x_1)=c_1\{\sigma(x_2)-\sigma(x_1)\};$$

therefore the function $\sigma(x)$ does not decrease on $[a,2a]$. Noting that $g(0)=\ln\varphi_1(0)=0$, from (8) we obtain

$$\ln c_1=\sigma(-\infty)-\sigma(\infty),$$

and consequently the function $\varphi_1(t)$ has the form (1).

5°. Because of lack of space, we omit the proofs of Theorems 1 and 2. We note that when $b<2a$, Theorem 1 is a special case of Theorem 3.

6°. Remarks on Theorem 1.

1) Theorem 1 may be supplemented by the following assertion. If the Poisson spectrum of an infinitely divisible law F is concentrated on $[-b,-a]$ and $0<a<b\leq 2a<\infty$, then $F\in I_0$.

2) From Theorem 1 and the preceding remark it follows that every infinitely divisible law F can be represented in the form

$$F=F_1*F_2*F_3*\cdots,$$

where $F_k\in I_0$ $(k=1,2,3,\ldots)$. Indeed, if $\varphi(t)$ is the characteristic function of the law F, then

$$\varphi(t)=e^{i\alpha t-\gamma t^2}\prod_{k=-\infty}^{\infty}\varphi_k^{(1)}(t)\varphi_k^{(2)}(t),$$

where α and γ are the quantities appearing in the representation of $\varphi(t)$ by Lévy' s formula, and $\varphi_k^{(1)}(t)$ $(\varphi_k^{(2)}(t))$, $k=0,\pm1,\pm2,\ldots$, are the characteristic functions of an infinitely divisible law with Poisson spectrum contained in $[-2^{k+1},-2^k]$ $([2^k,2^{k+1}])$; moreover, the corresponding function $M_1(x)$ $(M_2(x))$ in Lévy' s formula for $\varphi_k^{(1)}(t)$ $(\varphi_k^{(2)}(t))$, for $x\in[-2^{k+1},-2^k]$ $([2^k,2^{k+1}])$, is taken equal to the function $M_1(x)$ $(M_2(x))$ corresponding to $\varphi(t)$.

3) The condition $b\leq 2a$ in Theorem 1 is essential. Using the method of G. Cramér's paper $({}^5)$, it is not difficult to show that, for $0\leq a\leq 2a<b<\infty$, an infinitely divisible law for which

$$M_2'(x)\geq\text{const}>0\quad\text{for } a\leq x\leq b$$

does not belong to I_0.

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Note: Figure translations are in progress. See original paper for figures.