Reports of the Academy of Sciences of the USSR

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1965

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1965. Vol. 163, No. 6

GEOPHYSICS

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INVESTIGATION OF THE NONUNIQUENESS IN DETERMINING, FROM THE HODOGRAPH, THE PROPAGATION VELOCITY OF A SEISMIC WAVE

(Presented by Academician E. K. Fedorov on 6 February 1965)

I. Let a positive, piecewise twice differentiable function $u(y)$ be defined on the half-axis $y \geq 0$; let $u(0) = 1$, let $u(y)$ be bounded on every finite interval, and unbounded on the whole half-axis.

1. Consider the following problem (see Fig. 1). A material point is at the initial moment at the point O of the plane x, y. It begins to move in a direction forming an angle α with the y-axis, $0 < \alpha < \pi/2$. The trajectory of motion L is determined by the following conditions: 1) denote $\sin\alpha$ by p; at each point (x, y) of its descending branch the trajectory L forms with the y-axis an angle $\alpha(y) \leq \pi/2$ such that $\sin\alpha(y) = pu(y)$; 2) the deepest point of L has ordinate $Y(p) = \inf\{y,\ pu(y) \geq 1\}$; 3) if the abscissa of the deepest point $X(p) < \infty$, then L also has an ascending branch, symmetric to the descending one with respect to the straight line $x = X(p)$. Being at the point $(x, y) \in L$, the material point has velocity $u(y)$. Let $T(p)$ be the time of motion along L from 0 to $(X(p), Y(p))$. The trajectory L will be called a ray, and the number p the ray parameter. A ray with parameter p that has an ascending branch ends at the point $x = 2X(p),\ y = 0$; the time of motion along it is equal to $2T(p)$.

Fig. 1

The problem is posed as follows: the curve $\Gamma\{x = 2X(p),\ t = 2T(p),\ p \in (0, 1)\}$ (we shall call it the hodograph) is known; find $u(y)$.

2. The geophysical problem of determining the velocity section from the hodograph, more precisely an idealized version of this problem, can be reduced to this problem: the Earth is regarded as a sphere; it is assumed that seismic impulses propagate along seismic rays according to the laws of

Fig. 3

Figure 1: Fig. 3

geometrical optics, and that their propagation velocity depends only on depth.

Let us carry out this reduction. It is clearly sufficient to consider seismic rays not in a sphere, but in a disk. Let K be a disk of radius R with center C, and let γ be its circumference (see Fig. 2). Suppose that an impulse arises at a point $A \in \gamma$. Take polar coordinates r, θ, and let us measure angles from the radius CA. Let the propagation velocity of the impulse be $v(r)$, $r \in [0, R]$. The travel time to a point $B \in \gamma$ is equal to $\int_{AB} \frac{ds}{v}$, where

Fig. 2

$$ds = \sqrt{dr^2 + r^2 d\theta^2}.$$

If one makes the transformation

$$x = \frac{R}{v(R)}\theta, \qquad y = \frac{R}{v(R)} \ln \frac{R}{r},$$

$$u(y) = \frac{v\left(Re^{-(v(R)/R)y}\right)}{v(R)e^{-(v(R)/R)y}},$$

then we arrive at the problem on the plane described above.

(We assume that the angular distance between A and B along any ray AB

known exactly, and not modulo 2π *. The choice of the function $u(y)$ is determined by the following requirements: 1) let L denote the image of the ray AB; then

$$\int_{AB} \frac{ds}{v} = \int_{L} \frac{dl}{u}, \qquad dl = \sqrt{dx^2 + dy^2};$$

2) $u(0) = 1$.) The rays L considered in item I, 1 are, to be sure, only part of the images of the seismic rays AB: where the velocity has a discontinuity, the ray may split into a reflected and a refracted ray; in this case we consider only the refracted rays.

Fig. 3

Taking into account the remarks made, the mechanical problem I, 1 and the geophysical problem I, 2 are equivalent.

3. Natural from the physical point of view is the following additional restriction on $u(y)$: the set of points $y^0 > 0$ where

$$u(y^0) < s(y^0) = \sup\{u(y),\ y \in [0, y^0]\}$$

consists of a finite number of intervals. We shall assume that Γ is not arbitrary, but serves as the hodograph for some function $u(y)$ satisfying all the requirements of item I and the last restriction, and, moreover, that the function $X(p)$ (unknown to us in advance) is not constant on any segment.

II. Let us clarify some properties of the functions $Y(p)$, $X(p)$, and $T(p)$.

1a. Put $f(y) = 1/s(y)$. The function $f(y)$, obviously, does not increase, $0 < f(y) \leq 1$. Consider the graph of $f(y)$ in the plane y, p (see Fig. 3). Let y^0 be a point of discontinuity of $f(y)$; join the points $\{y^0, f(y^0 - 0)\}$ and $\{y^0, f(y^0 + 0)\}$ by a straight-line segment and adjoin it to the graph of $f(y)$. We proceed similarly with the graph of $Y(p)$. Then, by virtue of the relation

$$Y(p) = \inf\{y,\ pu(y) \geq 1\} = \inf\{y,\ f(y) \leq p\},$$

the graphs of $f(y)$ and $Y(p)$ (as curves in the plane y, p) coincide (on an interval of monotonicity these are mutually inverse functions).

b. The function $Y(p)$ does not increase. If for some p, $Y(p - 0) > Y(p + 0)$, then on the interval $(Y(p + 0), Y(p - 0))$ $f(y)$ is constant and equal to p. The latter condition is necessary and sufficient for the point $p \in (0, 1)$ to be a point of discontinuity of $Y(p)$.

From the assumption on the structure of the set of points y where $u(y) < s(y)$, it follows that there exist numbers $p_1 > p_2 > \cdots > p_n > 0$ possessing the following properties: put

$$y_k = Y(p_k + 0), \qquad \bar{y}_k = Y(p_k - 0), \qquad 1 \leq k \leq n;$$

then each of the intervals $j_k = (y_k, \bar{y}_k)$ contains points where $f(y) < 1/u(y)$, and outside the intervals j_k, $f(y) = 1/u(y)$. For definiteness we shall assume that $p_1 < 1$, so that for $y \in [0, y_1]$, $f(y) = 1/u(y)$. Put

$$p_0 = 1, \qquad p_{n+1} = 0, \qquad y_0 = \bar{y}_0 = 0.$$

We shall soon see that the numbers $p_1, \ldots, p_n$ play a very important role.

2. Mark on the interval $(0, 1)$ of the p-axis a finite number of points: the points $f(y - 0)$ and $f(y + 0)$, if $f(y - 0) > f(y + 0)$, and the points $f(y^0)$, if $f(y)$ is continuous but $f'(y)$ or $f''(y)$ either does not exist or is discontinuous at $y = y^0$. Divide the remaining points $p \in (0, 1)$ into two sets Π and Π', assigning to Π' those of them for which $f'(Y(p)) = 0$. Using the obvious relations

$$X(p) = \int_0^{Y(p)} \tan \alpha(y)\, dy = \int_0^{Y(p)} \frac{p\, dy}{\sqrt{u^{-2}(y) - p^2}},$$

* In the opposite case we arrive at the following generalization of problem I, 1: instead of the hodograph a curve $\tilde{\Gamma}$ is given $\{x = 2X(p),\ t = 2T(p),\ p \in (0,1)\}$, where $0 \le X(p) < \pi R/v(R)$, $X(p) \equiv X(p) \pmod{\pi R/v(R)}$. Denote by Γ^* the spatial curve $\{x = 2X(q),\ t = 2T(q),\ p = q;\ q \in (0,1)\}$, for which $\tilde{\Gamma}$ serves as the projection onto the plane x, t. Determining the hodograph from $\tilde{\Gamma}$ is done nonuniquely. But in the important case for applications, when Γ^* contains a finite number of components (the points (x, t, p) and $(x + 2\pi R/v(R), t, p)$ are identified), there exists only a finite number of hodographs Γ corresponding to $\tilde{\Gamma}$.

$$T(p) = \int_0^{Y(p)} \frac{dy}{u(y)\cos\alpha(y)} = \int_0^{Y(p)} \frac{dy}{u^2(y)\sqrt{u^{-2}(y) - p^2}},$$

one can show that for $p \in \Pi$, $X'(p)$ and $T'(p)$ exist and are continuous, and $T'(p) = pX'(p)$; for $p \in \Pi'$, $X(p) = X(p \pm 0) = T(p \pm 0) = T(p) = +\infty$. Let Π_0 be the subset of Π where $X'(p) = 0$; Π_0 is closed and, according to I, 3, is nowhere dense in Π.

One can prove the following property of the hodograph Γ. Let $(x_0, t_0) \in \Gamma$, and let Γ contain a smooth arc γ passing through (x_0, t_0) at an angle φ to the x-axis; if $p = \operatorname{tg}\varphi \in \Pi \cup \Pi'$, then $2X(p) = x_0$, $2T(p) = t_0$. Thus, from the given hodograph one can determine $X(p)$ and $T(p)$ for all $p \in \Pi \setminus \Pi_0$, and then (by continuity) also for $p \in \Pi_0 \cup \Pi'$. Hence, $X(p)$ and $T(p)$ are uniquely determined for all $p \in (0,1)$ (except, perhaps, for a finite number of points).

III, 1. Introduce the following notation:

$$\Phi(q) = \frac{2}{\pi}\int_q^1 \frac{X(p)\,dp}{\sqrt{p^2 - q^2}}, \qquad \tau(q) = \int_0^{Y(q)} \sqrt{u^{-2}(y) - q^2}\,dy, \qquad q \in (0,1);$$

$$\Psi(q) = \sum_{i=1}^{k} \frac{2}{\pi}\int_{y_i}^{\bar{y}_i} \operatorname{arctg}\sqrt{\frac{u^{-2}(y) - p_i^2}{p_i^2 - q^2}}\,dy,$$

$$\sigma_k = \int_{y_k}^{\bar{y}_k} \sqrt{u^{-2}(y) - p_k^2}\,dy, \qquad q \in (p_{k+1}, p_k), \quad 1 \le k \le n.$$

Below, along with $u(y)$, $Y(q)$, y_k, and $\bar{y}_k$, the functions $u^*(y)$, $Y^*(q)$ and the numbers y_k^* and $\bar{y}_k^*$ are considered. The expressions obtained from $\tau(q)$, $\Psi(q)$, and σ_k by replacing $u(y)$, $Y(q)$, y_k, and $\bar{y}_k$ with $u^*(y)$, $Y^*(q)$, y_k^*, and $\bar{y}_k^*$ are denoted by $\tau^*(q)$, $\Psi^*(q)$, and σ_k^*.

2. It is clear that $\varepsilon(p)$ has jumps σ_k at the points p_k, $1 \le k \le n$, and is continuous at the remaining points $p \in (0,1)$ (see II, 16). Since $\tau(p) =$

$T(p) - pX(p)$, the numbers p_k and σ_k are known. Since $\tau'(p) = -X(p)$, specifying Γ is equivalent to specifying $\tau(p)$.

3. We show that $Y(p) = \Phi(p) + \Psi(p)$, $p \in (0,1)$. Let D_q, D_q^0, and D^k be the following sets in the y,p-plane: $\{0 < y < Y(q),\ q < p < u^{-1}(y)\}$, $\{0 < y < Y(q),\ q < p < f(y)\}$, and $\{y \in \bar{j}_k,\ p_k < p < u^{-1}(y)\}$ (see Fig. 3); $F(p,y,q) = \dfrac{2p}{\pi}[(u^{-2}(y) - p^2)(p^2 - q^2)]^{-1/2}$. For $q \in (p_{k+1}, p_k)$, $0 \le k \le n$, we have

$$Y(q) = \int_0^{Y(q)} \left[\int_q^{u^{-1}(y)} F(p,y,q)\,dp\right] dy = \int_{D_q} F\,dS =$$

$$= \int_{D_q^0} F\,dS + \sum_{i=1}^{k} \int_{D^i} F\,dS = \Phi(q) + \Psi(q);$$

in particular, for $q \in (p_1, 1)$, $Y(q) = \Phi(q)$.

4. Suppose it has been possible to choose intervals $j_k^* = (y_k^*, \bar{y}_k^*)$, $1 \le k \le n$, and to specify $u^*(y)$ on them so that $\sigma_k^* = \sigma_k$, the function $Y^*(q) = \Phi(q) + \Psi^*(q)$ is nonincreasing, and $Y^*(p_k + 0) = y_k^*$, $Y^*(p_k - 0) = \bar{y}_k^*$ for any $k = 1, \dots, n$. From $Y^*(q)$ we determine $u^*(y)$ outside j_k^* so that $Y^*(p) = \inf\{y,\ pu^*(y) \ge 1\}$. It can be proved that $\tau^*(y) = \tau(y)$, i.e., $u^*(y)$ is a solution; moreover, it is clear that any solution can be constructed in this way.

IV, 1. If $y_i^* < y_i^* + h_i < \bar{y}_i^*$, $u^*(y) = u_i = \text{const}$ for $y \in (y_i^*,\ y_i^* + h_i)$ and $u^*(y) = p_i^{-1}$ for $y \in (y_i^* + h_i,\ \bar{y}_i^*)$, $1 \le i \le n$, then $\Psi^*(q)$ is given—

depends only on h_i (and not on $y_i^*, \bar{y}_i^*$). If, in addition, $h_i\sqrt{u_i^{-2} - p_i^2} = \sigma_i$ and the h_i are small, then $\Psi^*(q) < \Psi(q)$, and $Y^*(q) = \Phi(q) + \Psi^*(q)$ does not increase on $(0,1)$. Put $y_k^* = Y^*(p_k + 0)$, $\bar{y}_k^* = Y^*(p_k - 0)$, and define $u^*(y)$ from $Y^*(q)$ outside $\bar{j}_k^*$.

By virtue of III, 4, $u^*(y)$ is a solution. For different sets $(h_1, \dots, h_n)$, the solutions $u^*(y)$ are different. Note that for any $\varepsilon > 0$, for sufficiently small h_i,

$$0 < Y^*(q) - \Phi(q) < \varepsilon.$$

2. Let $u^0(y)$ and $u^1(y)$ be any two solutions; $Y^i(p) = \inf\{y,\ pu^i(y) \ge 1\}$, $i = 0, 1$. Fix arbitrarily $\omega \in (0,1)$. Put

$$Y^\omega(p) = (1-\omega)Y^0(p) + \omega Y^1(p),$$

$y_k^\omega = Y^\omega(p_k + 0)$, $\bar{y}_k^\omega = Y^\omega(p_k - 0)$,

$$s_k^\omega = (1-\omega)Y^0(p_k - 0) + \omega Y^1(p_k - 0);$$

Fig. 4

Figure 2: Fig. 4

$$u^\omega(y) = u^0\left[\frac{y - y_k^\omega}{1-\omega} + y_k^0\right] \quad \text{for } y_k^\omega < y < s_k^\omega,$$

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$$u^\omega(y) = u^1\left[\frac{y - s_k^\omega}{\omega} + y_k^1\right] \quad \text{for } s_k^\omega < y < \bar{y}_k^\omega;$$

outside the intervals $(y_k^\omega, \bar{y}_k^\omega)$, define $u^\omega(y)$ by $Y^\omega(p)$. According to III, 4, $u^\omega(y)$ is a solution for any $\omega \in [0,1]$.

Fig. 4

3. Put

$$F_k^i(r) = \operatorname{mes}\{y,\ y \le Y^i(p_k+0),\ u^i(y) \le r\}, \quad i = 0, 1, \quad 1 \le k \le n+1.$$

From III, 3 it follows: if $F_{k+1}^0(r) = F_{k+1}^1(r)$ for $r \le p_k^{-1}$, then $Y^0(p) = Y^1(p)$ for $p \in (p_{k+1}, p_k)$.

Note that the converse is also true: if, for some segment $[a,b] \subset (p_{k+1}, p_k)$, for $p \in [a,b]$ one has $Y^0(p) = Y^1(p)$, then $F_{k+1}^0(r) = F_{k+1}^1(r)$ for $r \le p_k^{-1}$.

4. Let $\{u_\nu(y)\}$ be the set of all solutions;

$$Y_\nu(p) = \inf\{y,\ pu_\nu(y) \ge 1\}.$$

Put

$$V(p) = \sup Y_\nu(p), \qquad N(p) = \inf Y_\nu(p).$$

By virtue of III, 3 and IV, 1, $N(p) = \Phi(p)$, $p \in (0,1)$. Clearly, $Y_\nu(p) \le T(p)/p$; therefore

$$V(p_k - 0) \le \inf\{T(p)/p,\ p < p_k\} < \infty$$

for every $k = 1, \dots, n$. Put $H_k = V(p_k - 0) - N(p_k - 0)$. It is not difficult to verify that

$$V(p) \le N(p) + H_k \quad \text{for } p \in (p_{k+1}, p_k).$$

If, in addition, $N(p) + \Psi^*(p)$ does not increase, then

$$V(p) = N(p) + \Psi^*(p).$$

V. Assign the point (y,u) to the set G (see Fig. 4) if

$$1 \le u \le p_1^{-1}, \qquad Y(u^{-1}+0) \le y \le Y(u^{-1}-0),$$

or if

$$u > p_1^{-1}, \qquad N(u^{-1}) < y < V(u^{-1}),$$

or if

$$0 < u \le p_k^{-1}, \qquad N(p_k + 0) < y < V(p_k - 0).$$

It follows from IV that:

1) for any point $(y^0, u^0) \in G$ there exists $u_\nu(y)$ such that $u_\nu(y^0) = u^0$;
2) for any $u_\nu(y)$ and for any $y \ge 0$, the point $(y, u_\nu(y)) \in \bar{G}$.

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Received
6 II 1965

Note: Figure translations are in progress. See original paper for figures.

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