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Abstract

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ON THE COMPUTATION OF HEAT-CONDUCTION AND GAS-DYNAMICS EQUATIONS BY SWEEPING OVER SEPARATE REGIONS

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In the numerical solution of multidimensional boundary-value problems for the equations of gas dynamics and heat conduction, the information pertaining to the next time step cannot always be placed in the machine' s operative memory. This circumstance makes it difficult to apply the sweep method for solving implicit difference equations, since one must too often resort to external memory devices. Below, using one-dimensional problems as an example, a method is indicated that makes it possible to divide the boundary-value problem into several smaller problems, each of which can be solved by known methods (see, for example, ($^{1-3}$)).

We shall devote primary attention to the stability of the proposed computational method; it is therefore natural to restrict ourselves to linear equations and linear homogeneous boundary conditions.

Consider, in the region $[t \geq 0;\ -l_1 \leq x \leq l_2]$, the problem for the heat-conduction equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}; \qquad a = \begin{cases} a_1, & \text{if } x < 0, \\ a_2, & \text{if } x > 0 \end{cases} \tag{1}$$

$(a_1, a_2 = \text{const})$ with boundary conditions

$$\begin{gathered} \alpha_1 u + \beta_1 \partial u/\partial x = 0 \quad (x = -l_1); \qquad \alpha_2 u + \beta_2 \partial u/\partial x = 0 \quad (x = l_2); \\ u_{-0} = u_{+0}; \qquad a_1 \partial u/\partial x\big|_{x=-0} = a_2 \partial u/\partial x\big|_{x=+0} \end{gathered} \tag{2}$$

and initial condition $u(x, 0) = \varphi(x)$. (In the case $a_1 = a_2$, the boundary conditions at $x = 0$ are superfluous.)

For the numerical solution of this problem we apply the following difference scheme:

$$
\begin{gathered}
u_{1,i}^{k+1}-u_{1,i}^{k}=\left(u_{1,i+1}^{k+1}-2u_{1,i}^{k+1}+u_{1,i-1}^{k+1}\right)a_1\tau/h_1^2 \quad (i=-n_1+1,\dots,1),\\
\alpha_1 u_{1,i}^{k+1}+\beta_1\left(u_{1,i+1}^{k+1}-u_{1,i}^{k+1}\right)/h_1=0 \quad (i=-n_1),\\
a_1\left(u_{1,0}^{k+1}-u_{1,-1}^{k+1}\right)/h_1=a_2\left(u_{2,1}^{k}-u_{2,0}^{k}\right)/h_2;
\end{gathered} \tag{3}
$$

$$
\begin{gathered}
u_{2,i}^{k+1}-u_{2,i}^{k}=\left(u_{2,i+1}^{k+1}-2u_{2,i}^{k+1}+u_{2,i-1}^{k+1}\right)a_2\tau/h_2^2 \quad (i=1,\dots,n_2-1),\\
\alpha_2 u_{2,i}^{k+1}+\beta_2\left(u_{2,i}^{k+1}-u_{2,i-1}^{k+1}\right)/h_2=0 \quad (i=n_2),\\
u_{2,0}^{k+1}=u_{1,0}^{k+1},
\end{gathered} \tag{4}
$$

i.e., we shall transmit, as the boundary condition from region 2 into region 1, the heat flux, and from region 1 into region 2—the temperature.

This scheme makes it possible to carry out the transition from the k-th to the $(k+1)$-st step by means of two independent sweeps. First, using the boundary values $u_{2,1}^{k}$ and $u_{2,0}^{k}$, we solve system (3). Then, taking from the obtained solution the boundary value $u_{1,0}^{k+1}$, we solve system (4).

The computations carried out showed that in those cases in which scheme (3), (4) is stable, it gives good results, comparable in accuracy with the results of a through sweep over both regions.

Let us now consider the question of the stability of the scheme (3), (4). In doing so, we shall confine ourselves to obtaining only necessary stability conditions, and only for very large and very small values of the quantity $c=a\tau/h^2$. To simplify the notation, in what follows we shall omit the indices of the regions (1 and 2) wherever they are immaterial or where the consideration is being carried out simultaneously for both regions.

Let us estimate the largest, in modulus, eigenvalue of the linear transformation $u^k\to u^{k+1}$ effected by the scheme (3), (4). Setting $u_i^{k+1}=\lambda u_i^k$ and substituting this expression into the first equations of the systems (3) and (4), we obtain second-order difference equations, so that the solutions of these equations have the form

$$u_i^k=\xi\mu^i+\psi\mu^{-i},$$

where μ is one of the roots of the equation

$$\mu^2-2\left(1+\frac{\lambda-1}{2\lambda c}\right)\mu+1=0, \tag{5}$$

and ξ and ψ are to be determined from the boundary conditions. Obviously, one may assume that $|\mu|\leqslant 1$; moreover, from the form of equation (5) it follows that when $|\mu| = 1$, λ is real and $|\lambda| \leqslant 1$, and therefore it is sufficient to restrict ourselves to the case $|\mu| < 1$.

Eliminating ξ_1, ξ_2 and ψ_1, ψ_2 by means of the boundary conditions (the second and third equations of the systems (3) and (4)), we obtain

$$\lambda = -\frac{a_2 h_1}{a_1 h_2} \frac{1-\mu_2}{1-\mu_1} + O(\mu^{2n-1}). \tag{6}$$

As $c \to 0$, it follows from (5) that $\mu \to 0$, i.e. $\lambda \to -(a_2 h_1)/(a_1 h_2)$. Thus, the necessary stability condition has the form

$$a_1/h_1 > a_2/h_2 \qquad (c \to 0). \tag{7}$$

As $c \to \infty$, equation (5) has the solution

$$\mu = 1 - \sqrt{\frac{\lambda - 1}{\lambda c}} + O\left(\frac{\lambda - 1}{\lambda c}\right),$$

so that

$$\lambda = -\sqrt{a_2/a_1} + O\left(\frac{1}{c}\right) + O(\mu^{2n-1}). \tag{8}$$

A special feature of this case is that, as $c \to \infty$, $\mu \to 1$, so that, in general, the quantity $O(\mu^{2n-1})$ cannot be neglected. However, as is easy to prove, if $\tau \to 0$, then $n \to \infty$ sufficiently rapidly for $\mu^{2n-1} \to 0$. Thus, we obtain the second necessary stability condition

$$a_1 \geqslant a_2 \qquad (c \to \infty). \tag{9}$$

In the computation of specific problems, examples were found in which the accuracy of the solution sufficient for the given problem was obtained with such time steps τ that did not ensure smallness of the third term in expression (8). In this connection we indicate one more condition, ensuring nonincrease of the solution in the case of finite steps τ; as $\tau \to 0$, this condition passes into (9).

Passing from the difference equations (3), (4), as $c \to \infty$, to the differential equation $(\lambda - 1)u = \lambda a \tau u''$. Writing down, analogously to what we did for the difference equation, the general solution and taking the boundary conditions into account, we obtain

$$|\lambda| \equiv \left| -\sqrt{\frac{a_2}{a_1}} \frac{\alpha_1(1-\xi_1) + \beta_1(1+\xi_1)\sigma_1}{\alpha_1(1+\xi_1) + \beta_1(1-\xi_1)\sigma_1} \frac{\alpha_2(1+\xi_2) - \beta_2(1-\xi_2)\sigma_2}{\alpha_2(1-\xi_2) - \beta_2(1+\xi_2)\sigma_2} \right| \leqslant 1, \tag{10}$$

where $\sigma = -\sqrt{(\lambda-1)/\lambda a\tau}$, $\xi = \exp(2\sigma l)$.

The computations carried out showed that, in the presence of instability, the largest eigenvalue in modulus turns out to be real negative.

···finite. Therefore it is natural to substitute into estimate (10) the value $\lambda = -1$:

$$\sigma = -\sqrt{2/a\tau}, \qquad \xi = \exp(-3/\sqrt{c_0}), \qquad c_0 = a\tau/l^2. \tag{11}$$

Calculations showed that simultaneous fulfillment of conditions (7) and (10), (11) ensures stable computation and (with a reasonable choice of τ) sufficient accuracy. As $\tau \to 0$, condition (10) is weakened and passes into (9). Thus, simultaneous fulfillment of conditions (7) and (9) proves to be a sufficient condition for the stability of the difference scheme (3), (4) for any fixed value of c (however, numerical computation without overflow is then possible only for such $\tau \le \tau_0$ that $c_0 = a\tau_0/l^2 \ll 1$).

Let us now consider the system of gas-dynamics equations

$$\partial u/\partial t + v_0 \partial p/\partial x = 0,$$

$$\partial v/\partial t - v_0 \partial u/\partial x = 0, \qquad p = Av^{-\gamma}, \tag{12}$$

and take for it the difference scheme

$$u_i^{k+1} - u_i^k + \frac{1}{2}\left(p_{i+1/2}^{k+1} - p_{i-1/2}^{k+1}\right)\left(v_{0,i-1/2} + v_{0,i+1/2}\right)\tau/h = 0$$

$$(i = -n_1, \dots, -1, 1, \dots, n_2),$$

$$v_{i+1/2}^{k+1} - v_{i+1/2}^{k} - \left(u_{i+1}^{k+1} - u_i^{k+1}\right) v_{0,i+1/2}\tau/h = 0,$$

$$p_{i+1/2}^{k+1} = A\left(v_{i+1/2}^{k+1}\right)^{-\gamma} \qquad (i = -n_1, \dots, n_2 - 1),$$

$$p_{-1/2}^{k+1} = p_{1/2}^{k}. \tag{13}$$

The last of the boundary conditions makes it possible to carry out a sweep on the interval $[-l_1, 0]$ (by a sweep for a nonlinear system we mean solving this system by Newton' s method, with the solution of the linear system carried out by the usual sweep). After this, taking the obtained value u_0^{k+1} as the boundary condition at $x = 0$, we carry out the sweep on the interval $[0, l_2]$.

The necessary stability conditions for the scheme obtained by linearizing scheme (13), under the assumption that its coefficients are constant, can be found by the same method as for the heat-conduction equation. They have the form

$$\frac{v_1}{v_{01}}\frac{h_1}{\gamma_1} \ll \frac{v_2}{v_{02}}\frac{h_2}{\gamma_2} \qquad \left(s\frac{\tau}{h} \to 0\right); \tag{14}$$

$$\frac{s_1}{\gamma_1} \ll \frac{s_2}{\gamma_2}, \qquad s = \sqrt{\gamma p v} \qquad \left(s\frac{\tau}{h} \to \infty\right). \tag{15}$$

Just as in the case of the heat-conduction equation, condition (13) does not ensure computation without overflow for finite time steps τ. The expression corresponding to (10) can be obtained by the same method. Its analysis shows that condition (15) is practically sufficient if $s(\tau/l) \ll 1$.

The calculations performed confirmed the essential nature of the necessary conditions obtained. In practical use of the proposed method it is convenient to place the division points on the boundaries of different substances. In this case conditions (9) and (15) are fulfilled automatically (with the corresponding numbering of the regions), and inequalities (7) and (14) are easily satisfied by selecting the grid. If inequalities (9) and (15) are satisfied with a large margin, then checking condition (10) and the corresponding condition for the gas-dynamics equations becomes superfluous for practically used time steps.

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Note: Figure translations are in progress. See original paper for figures.

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