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Abstract

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ON THE COMPLETENESS OF A SYSTEM OF EIGEN- AND ASSOCIATED ELEMENTS OF NON-SELF-ADJOINT OPERATORS

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Let B be a completely continuous operator possessing a complete system of eigen- and associated (e.a.) elements forming a basis in the Hilbert space $\mathcal{H}$. Consider the operator

$$L(\lambda) = A_0 + \lambda A_1 B + \cdots + \lambda^{n-1} A_{n-1} B^{n-1} + \lambda^{nB^{n.}}$$

In the present note one sufficient condition is established for the completeness of the system of e.a. elements of the operator $L(\lambda)$. Along the way, some properties of the resolvents $(E - \lambda B)^{-1}$ and $(E - L(\lambda))^{-1}$ are studied.

In what follows $y_{i,k}$ $(k = 0, 1, \dots, n_i)$ will denote an eigen-element corresponding to the eigenvalue λ_i for $k = 0$, and the corresponding associated elements for $k = 1, 2, \dots, n_i$, $a_{i,j}(f) = (f, z_{i,j})$, where $z_{i,k}$ is the system biorthogonal to $y_{i,k}$. In the work ([1]) of M. V. Keldysh it is shown that $z_{i,n}$ is a certain system of e.a. elements of the adjoint operator B^*.

Lemma 1. *The resolvent of the operator B has the form*

$$(E - \lambda B)^{-1} f = \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \left(\sum_{k=j}^{n_i} a_{i,k} A_{i,j}^k \right) y_{i,j},$$

where $A_{i,j}^k$ are found from the recurrence relations

$$A_{i,k}^k = \frac{\lambda_i}{\lambda_i - \lambda} \equiv \varphi_i(\lambda) \quad \text{for all } k \text{ and } i,$$

$$A_{i,j}^k = \sum_{l=j}^{k-1} (-1)^{k-l-1} A_{i,j}^l \varphi_i(\lambda) \frac{\lambda}{\lambda_i^{k-l+1}}. \tag{1}$$

Proof. Let

$$f = \sum_{i=1}^{\infty} \sum_{k=0}^{n_i} a_{i,k} y_{i,k};$$

then

$$(E - \lambda B)^{-1} f = \sum \sum a_{i,k} (E - \lambda B)^{-1} y_{i,k}.$$

Since

$$y_{i,k} = \lambda_i B y_{i,k} + B y_{i,k-1}, \quad k = 1, 2, \dots, n_i,$$

and $y_{i,0} = \lambda_i B y_{i,0}$, it follows that

$$B y_{i,k} = \frac{1}{\lambda_i} y_{i,k} - \frac{1}{\lambda_i} B y_{i,k-1}; \qquad B y_{i,0} = \frac{1}{\lambda_i} y_{i,0}.$$

Hence we easily obtain that

$$B y_{i,k} = \sum_{j=1}^{k+1} (-1)^{j+1} \frac{1}{\lambda_i^j} y_{i,k-j+1}$$

and, consequently,

$$(E - \lambda B)^{-1} y_{i,k} = y_{i,k} - \lambda \sum_{j=1}^{k+1} (-1)^{j+1} \frac{1}{\lambda_i^j} y_{i,k-j+1}.$$

Applying $(E - \lambda B)^{-1}$ and solving with respect to $(E - \lambda B)^{-1} y_{i,k}$, we obtain

$$(E - \lambda B)^{-1} y_{i,k} = \varphi_i(\lambda) y_{i,k} + \varphi_i(\lambda) \lambda \sum_{j=2}^{k+1} (-1)^{j+1} \frac{1}{\lambda_i^j} (E - \lambda B)^{-1} y_{i,k-j+1}. \quad (2)$$

Since $(E - \lambda B)^{-1} y_{i,0} = \dfrac{\lambda_i}{\lambda_i - \lambda} y_{i,0} \equiv \varphi_i(\lambda) y_{i,0}$, it follows from (2) that $(E - \lambda B)^{-1} y_{i,k}$ must have the form

$$(E - \lambda B)^{-1} y_{i,k} = \sum_{j=0}^{k} A_{i,j}^k y_{i,j},$$

where $A_{i,j}^k$ depends on λ. To find $A_{i,j}^k$, let us turn to formula (2) and replace $(E-\lambda B)^{-1}y_{i,l}$ on the left- and right-hand sides by $\sum_{j=0}^{l} A_{i,j}^l y_{i,j}$; we obtain

$$\sum_{j=0}^{k} A_{i,j}^k y_{i,j} = \varphi_i(\lambda) y_{i,k} + \lambda\varphi_i(\lambda) \sum (-1)^{j+1} \frac{1}{\lambda_i^j} \left(\sum_{r=0}^{k-j+1} A_{i,r}^{k-j+1} y_{i,r} \right).$$

Comparing the coefficients of y_{ij} on the left- and right-hand sides, we obtain:

$$A_{i,j}^k = \lambda\varphi_i(\lambda) \sum_{l=j}^{k-1} (-1)^{k-l-1} A_{i,j}^l \frac{1}{\lambda_i^{k-l+1}}, \qquad j<k; \qquad A_{ij}^j = \varphi_i(\lambda).$$

This proves the lemma.

For what follows, let us note that $A_{i,j}^k$ are determined not by the indices k and j themselves, but by the difference $k-j$. Indeed:

$$A_{i,j}^k = \lambda\varphi_i(\lambda) \sum_{l=j}^{k-1} (-1)^{k-l-1} A_{i,j}^l \frac{1}{\lambda_i^{k-l+1}} = \lambda\varphi_i(\lambda) \sum_{s=1}^{k-j} (-1)^{s+1} A_{i,j}^{k-s} \frac{1}{\lambda_i^{s+1}}.$$

It is seen from this that if $A_{i,j}^{k-s}$, $s=1,2,\dots,k-j$, depend only on the difference $k-s-j$ (i.e. for $l=j,j+i,\dots,k-1$ depend on $l-j$), then $A_{i,j}^k$ depends only on $k-j$, and not on k and j. But since $A_{i,k}^k$ depends only on i, by induction $A_{i,j}^k$ will depend on i and $k-j$; therefore one may introduce the notation $A_{i,j}^k = B_{i,k-j} = B_{i,r}$.

Definition. We shall say that a ray belongs to the class $\mathcal{K}_\beta$ if it is the bisector of some angle of aperture not smaller than 2β, inside which there are only finitely many eigenvalues of the operator B.

Lemma 2. *If the orders of the eigenelements of the operator B are bounded in the aggregate (i.e. $n_i \le m$ for all i) and the system of root elements of the operator B forms a basis of Riesz type (a P-basis), then the resolvent of the operator $A+\lambda B$ on each ray from $\mathcal{K}_\beta$ has the form*

$$(E-A-\lambda B)^{-1} = (E+M(\lambda))(E-\lambda B)^{-1},$$

where $M(\lambda)$ on rays from $\mathcal{K}_\beta \to 0$ as $|\lambda| \to \infty$.

Proof.

$$Y = (A+\lambda B)y + f, \qquad Y = (E-\lambda B)^{-1}Ay + (E-\lambda B)^{-1} f.$$

Using the expression for $(E-\lambda B)^{-1}$, let us estimate $\|(E-\lambda B)^{-1}A\|$:

$$\|(E-\lambda B)^{-1}Af\| = \left\| \sum_{i=1}^{\infty} \sum_{j=0}^{n_i} \left(\sum_{k=j}^{n_i} a_{i,k}(Af) A_{i,j}^k \right) y_{i,j} \right\| \le$$

$$\leq \left\| \sum_{i=1}^{N} \sum_{j=0}^{n_i} \left(\sum_{k=j}^{n_i} a_{ik}(Af) A_{i,j}^k \right) y_{i,j} \right\| + \left\| \sum_{i=N+1}^{\infty} \sum_{j=0}^{n_i} \left(\sum_{k=j}^{n_i} a_{i,k}(Af) A_{i,j}^k \right) y_{i,j} \right\| \leq$$

$$\leq M\|A\| m \left(\sup_{\substack{i \leq N \\ j < n_i}} \sum |A_{i,j}^k|^2 \right)^{1/2} + Mm \left(\sup_{j,i} \sum_{k=j}^{n_i} |A_{i,j}^k|^2 \right) \|Af\|_{P_N \mathcal{H}},$$

where M is the constant appearing in the definition of a Riesz basis; $m = \max n_i$; P_N is the projection operator of the subspace generated by the elements $y_{i,k}$, $i \geq N+1$ (in general not orthogonal, but equivalent to an orthogonal one).

Since A is a completely continuous operator, $\|Af\|_{P_N \mathcal{H}}$ can be made arbitrarily small by choosing N sufficiently large ($\|f\| = 1$);

therefore for any $\varepsilon > 0$ one can find such an $N_0(\varepsilon)$ that, for $N \geq N_0(\varepsilon)$,

$$\|Af\|_{P_N \mathcal{H}} < \varepsilon, \qquad \|Af\|_{P_N \mathcal{H}} = \|P_{NAf}\|.$$

We shall prove that if $\lambda \in \mathcal{K}_\beta$, then

$$\sup_{j,i} \sum_{i=j}^{n_i} |A_{i,j}^k|^2 < P,$$

where P is some number independent of λ, and

$$\lim_{|\lambda| \to \infty} \sup_{i \leq N,\, j \leq m} \sum |A_{i,j}^k|^2 = 0.$$

First we show that if $B_{i,r} = \varphi_i(\lambda) O(1)$, $r \leq t$, then $B_{i,t+1} = \varphi_i(\lambda) O(1)$. Indeed:

$$B_{i,r+1} = \lambda \varphi_i(\lambda) \sum_{s=1}^{r+1} (-1)^{s+1} B_{i,r-s+1} \frac{1}{\lambda_i^{s+1}},$$

$$|B_{i,r+1}| \leq |\varphi_i| \left(\sum \left| \frac{\lambda}{\lambda_i} \varphi_i(\lambda) \right| |O(1)| \left| \frac{1}{\lambda_i^s} \right| \right) \leq |\varphi_i| O(1),$$

since

$$\frac{\lambda}{\lambda_i} \varphi_i(\lambda) = \frac{\lambda}{\lambda_i - \lambda} = \psi_i(\lambda); \qquad |\psi_i| \leq \frac{1}{\sin \beta}.$$

It follows that

$$|A_{i,j}^k|^2 \leq |\varphi_i(\lambda)|^2 O(1);$$

but since

$$\varphi_i(\lambda) = \frac{\lambda_i}{\lambda_i - \lambda},$$

we have

$$|\varphi_i| \leq \frac{1}{\sin\beta},$$

and if $i \leq N$, then

$$\lim_{|\lambda|\to\infty} |\varphi_i| = 0.$$

Taking into account the assumptions of the theorem, namely that k and j are bounded by the number m, and using the formula

$$(E - A - \lambda B)^{-1} = (E - (E - \lambda B)^{-1}A)^{-1}(E - \lambda B)^{-1},$$

we arrive at the assertion of Lemma 2. Indeed, if one sets

$$(E - (E - \lambda B)^{-1}A)^{-1} = E + M(\lambda),$$

then we have

$$\|(E - \lambda B)^{-1}A\| \leq Mm \left\{ \left(\sup \sum_{k=j}^{n_i} |A_{i,j}^k|^2 \right)^{1/2} \|A\| + \sup_{j,i} \sum_{k=j}^{n_i} |A_{ij}^k|^2 \|P_{NAf}\| \right\}.$$

Since for any $\varepsilon > 0$ one can find such an R that, for $|\lambda| > R$, the expression in braces does not exceed ε, we have

$$(E - (E - \lambda B)^{-1}A)^{-1} = \sum_{i=0}^{\infty} C_i(\lambda), \qquad \text{where } C(\lambda) = (E - \lambda B)^{-1}A.$$

Thus Lemma 2 is proved.

An analogous lemma is also true for the operator $L(\lambda)$.

We now prove the main theorem.

Theorem. If B is a completely continuous operator, $B \in \gamma_\rho$, whose eigen- and associated elements form a basis of Riesz type, and the class of rays $\mathcal{K}_\beta^n$ is ε-dense in G for $\varepsilon \leq \pi/\rho n$, $\beta \geq \beta_0 > 0$, then the system of eigen- and associated elements of the operator is n-fold complete in the space $\mathcal{H}$.

Proof. Consider the equation

$$y = L^*(\lambda)y + f,$$

where

$$f(\lambda) = \sum_{i=0}^{n-1} \lambda^i f_i$$

is chosen so that the element $f = \{f_0, \dots, f_{n-1}\}$ is orthogonal to the elements of the operator $L(\lambda)$; the solution $y(\lambda)$ must be an entire function of order not exceeding $n\rho$.

Since on the rays of $\mathcal{K}_\beta^n$ (a ray belongs to $\mathcal{K}_\beta^n$ if, upon rotation by $(n-1)\varphi$, where φ is the argument of the ray, it coincides with some ray from $\mathcal{K}_\beta$)

this function grows no faster than a polynomial of degree $n-1$, then on the basis of the Phragmén–Lindelöf theorem we conclude that $y(\lambda)$ is a polynomial of degree not higher than $n-1$:

$$y(\lambda) = \sum_{i=0}^{n-1} \lambda^i y_i.$$

We shall show that $y(\lambda) \equiv 0$. Suppose this is not so and y_{i_0} is the nonzero coefficient of the highest power of λ. We have:

$$y(\lambda) = \sum_{i=0}^{n-1} \lambda^i (A_{iB}^i)^* y(\lambda) + \lambda^n (B^i)^* y(\lambda) + f(\lambda).$$

Comparing the coefficients of equal powers of λ in the left- and right-hand sides, we obtain $(B^n)^* y_{i_0} = 0$.

$(B^{n*} y_{i_0}, g) = 0$ for any g. $(y_{i_0}, B^n g) = 0$ shows that y is orthogonal to all associated elements of the operator B^n, and these elements are complete in $\mathcal{H}$; hence it follows that $y_{i_0} = 0$; consequently, all $f_i = 0$. The theorem is proved.

Let now H be a complete self-adjoint operator having finite order ρ, and let A_i be completely continuous operators.

Theorem 2. *For any $0 \leq \alpha \leq 1$, the system of eigen and associated elements of the operator*

$$A_0 + \lambda H^{\alpha/n} A_1 H^{(1-\alpha)/n} + \dots + \lambda^{n-1} H^{(n-1)\alpha/n} A_{n-1} H^{(n-1)(1-\alpha)/n} + \lambda^{nH}$$

is n-fold complete in $\mathcal{H}$.

The proof of Theorem 2 is, in essence, close to the proof of Theorem 1.

We note that, analogously to how this was done in (2) for A_0, the conditions on A_i can be weakened by requiring that the norm of the purely bounded part be sufficiently small.

We shall give one more theorem concerning operators of the form

$$A_0 + \lambda H^{1/n} A_1 + \dots + \lambda^{n-1} H^{(n-1)/n} A_{n-1} + \lambda^{nH}.$$

For simplicity of formulation of the theorem, we shall assume that H is a positive operator, although in fact similar results are valid even if H is a normal operator whose eigenvalues lie inside certain angles.

Consider the equation $y = \lambda^{nHy}$. If H is a complete self-adjoint operator, then the eigenvalues $\lambda_{i,k}$ and μ_i of the equations $y = \lambda^{nHy}$ and $y = \mu H y$ are connected by the relation $\lambda_{i,k}^n = \mu_i$ $(k = 0, 1, \dots, n-1)$, i.e., the eigenvalues of the first equation lie on rays passing through the nth roots of unity. Take n nonintersecting angles with vertex at the origin such that the indicated rays lie strictly inside these angles, and for each number r form the domains $\Gamma_{\psi_i, r}$ $(i = 0, 1, \dots, n-1)$, consisting of the angle ψ_i and the circle of radius r with center at the origin.

Theorem 3. *Suppose that for some $0 < \rho < 1$ the following conditions are satisfied: either the operators $H^{-p} A_i$ $(i = 0, \dots, n-1)$ are bounded and $\lim_{k\to\infty} \dfrac{k}{\mu_k^p} = 0$, or else the operators $H^{-p} A_i$ $(i = 0, \dots, n-1)$ are completely continuous and $\lim_{k\to\infty} \dfrac{k}{\mu_k^p} < \infty$. Then for each set of numbers $\alpha_1, \dots, \alpha_j^{(j)}$, $\alpha_j \le n$, $\alpha_i \ne \alpha_k$, one can find an r such that the system of eigen and associated elements corresponding to eigenvalues lying in the domain*

$$\sum_{i=1}^{j} \Gamma_{\psi_i, r}$$

is j-fold complete in $\mathcal{H}$, and some subsequence of the partial sums of the corresponding expansion (see (1)) converges.

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References

[1] M. V. Keldysh, DAN, 77, No. 1 (1951). [2] Dzh. E. Allakhverdiev, DAN, 115, No. 2 (1957).

Note: Figure translations are in progress. See original paper for figures.